HURST TRADING WITH AN EXCURSION INTO FRACTAL SPACE OF RETURNS

A Dissertation

Presented to the Faculty of the Graduate School

of Cornell University

in Partial Fulfillment of the Requirements for the Degree of

Doctor of Philosophy

by

Paitoon Wongsasutthikul

January 2012

HURST TRADING WITH AN EXCURSION INTO FRACTAL SPACE OF RETURNS

Paitoon Wongsasutthikul, Ph.D.

Cornell University 2012

This dissertation tackles the problem of non-normality in the distribution of returns and attempts to formulate a proprietary trading strategy to arbitrage the markets using appropriate statistical and mathematical tools. The first essay provides fundamental understanding to fractional Brownian motion (fBm) process, its characteristic Hurst exponent, and the concept of unit root in time series data. The study shows that a simple autoregressive (AR) process with suitable lag coefficients is able to effectively replicate the fractal time series and preserves its characteristic Hurst exponent. More interestingly, an equation that defines the relationship between the AR lag coefficients and the Hurst exponent that described a particular fBm process is also derived.

The second essay introduces the concept of excursion measures and illustrates how the Itô's excursion theory can be used as a tool to understanding fractals. The excursions-valued process is shown to follow a binomial distribution which is a robust substitute for Poisson distribution as suggested from the theory. The results also show that a process with low Hurst exponent or short-memory process has higher mean excursion measure at

low excursion length as compared to a process with high Hurst exponent or a long-memory process. On the other hand, we see systematic wandering with longer excursion in a long-memory process with Hurst exponent higher than 0.5.

Based on the discovery from the first two essays, the third essay combines these findings together to form a trading strategy called “Hurst Trading” with trading signals generated from the fluctuation in the dynamics of the Hurst exponent across time, among other indicators. We find that for the period between 2002 to 2011 the Hurst Trading strategy is able to outperform the traditional momentum strategy and the “Buy and Hold” strategy by a wide margin on stock trading in the DJIA Index, SPX Index, and R2500 Index. Furthermore, the more fractal the process is, the higher the chance that the Hurst Trading algorithm would be able to correctly time the entry/exit points in the market.

BIOGRAPHICAL SKETCH

Paitoon Wongsasutthikul was born in Bangkok, Thailand to Sutthichai Wongsasutthikul and Bussaba Paisarnsukanant. He has two, younger brothers: Paiboon Wongsasutthikul and Pairoj Wongsasutthikul. He grew up in Bangkok and finished his high school at Ekamai International School in 2001. He received his Bachelor of Engineering degree with First Class Honor in Mechanical Engineering from Sirindhorn International Institute of Technology at Thammasat University in March of 2005. It was during his senior year at Thammasat University that Paitoon was selected to receive the Anandamahidol Foundation scholarship to pursue graduate studies in specialized field abroad. The Anandamahidol Foundation is a non-profit and non-governmental organization under The Royal Patronage of His Majesty King Bhumibol Adulyadej of Thailand.

Paitoon went to Stanford University in Palo Alto, California and earned a Master of Science degree in Management Science and Engineering in June of 2006. Subsequently, he began working on a sales and trading desk covering financial derivatives and structured products at ABN AMRO Bank N. V. in Hong Kong and Singapore. In August of 2008, Paitoon returned to school to complete his doctoral study in the Ph.D. program at the Dyson School of Applied Economics and Management at Cornell University in Ithaca, New York.

ACKNOWLEDGMENTS

This dissertation would not be completed without the support and guidance of the following individuals to whom I am deeply indebted. First, I would like to thank my special committee chair and thesis advisor, Calum Turvey. His wisdom, creativity, and encouragement have enlighten me and made this learning experience an invaluable one. He always supported me to think out of the box and pursue research topics that I found interesting. I also thank him for his gracious hospitality and friendship all throughout my years at Cornell. Additionally, I would like to thank the rest of my special committee members: David Ng, Vicki Bogan, and Edith Liu. They have given me so many valuable insights and comments relating to my research work. It has been a privilege working with this group of brilliant researchers.

This dissertation was made possible with the financial support from the Anandamahidol Foundation. Beyond funding, more important gifts were the inspiration and focus that I have developed over the years from being a King’s scholar. It has changed my life and shaped me into a better person. I am deeply indebted for the opportunity that has been given to me by the foundation.

Additionally, my deep gratitude goes out to all my friends at Cornell. It would be a totally different experience for me without your warm friendship. I am especially thankful to all my Thai friends who have been very kind and helpful in every ways possible. Thanks for welcoming me into your big family.

I also would like to thank Catherina Tan for her everlasting support and encouragement in both good and bad times. She gives me strength to fight this long journey and is always there for me. I deeply appreciate her confidence in me and her companionship throughout every day of this nearly half-decade process.

Finally, I would like to thank my family for their unconditional love and support throughout my life. My two brothers are ever supportive and understanding. They are always helpful and never let me down. I am very proud of them. Last, but not least, I am deeply indebted to my parents for their guidance and care for me. They are my endless source of support, encouragement, and happiness. I would like to thank them for being patient with me through all these years that I was away from home. Without them, I would never be the man I am today. The completion of this dissertation would not be possible at all without the support from my family. I love them dearly and appreciate all that they have done for me.

TABLE OF CONTENTS

CHAPTER I

INTRODUCTION

The Efficient Market Hypothesis (EMH) has been the core structure of financial economics theory for decades. EMH states that current market prices of securities fully reflect all available information and that price returns behave like a "nice" random walk. However, recent studies have found evidences against EMH and suggested that markets are indeed fractals. This implies that there is a lucrative opportunity for traders who could understand the underlying characteristics of the price process to make consistent profits in the market with proper trading model.

This dissertation employs a bottom-up approach in building a trading algorithm that seizes to make profits from the movements in the market. In order to exploit any forms of market movements, some first-principle understanding of what constitutes a stochastic process is necessary. The most basic of these principles lies in the real meaning of the unit root: the significance of the unit root to a stable stochastic process and how autoregressive processes with unit root behaves relative to a Brownian motion. The second fundamental principle is in the behavior of the stochastic processes themselves. Too many academic literatures on this subject focus almost entirely on the measurement of the unit root with virtually no study examining how Brownian processes operate in practice. To address this point, we need to understand how to model the intertemporal relationship and the random movements in asset price path. In other words, what does the pattern of excursions from a fixed point or a mean return say about the underlying properties of the stochastic process? The first two essays of this dissertation focus on the elemental aspects of these two questions. The third essay puts this knowledge into the overall problem of how we can exploit the arbitrage opportunities in the market by incorporating Hurst exponent in a trading mechanism.

In the first essay, we establish the linkage between an autoregressive process with unit root and a fractional Brownian motion that is characterized by a certain Hurst exponent. The key objectives of this essay can be summarized as follow.

Objective 1: To show the necessary and sufficient condition for the existence of a unit root in any autoregressive models.

Objective 2: To demonstrate how the scaled variance ratio technique can be applied to any time series data in order to estimate its characteristic Hurst exponent.

Objective 3: To derive an equation that defines the relationship between the AR lag coefficients and the Hurst exponent that characterized a particular fBm process.

Objective 4: To replicate a fractal time series and preserve its Hurst exponent measure using a combination of stepwise regression and restricted least square method.

Previously, various methods and techniques used to generate fBm have been implemented and utilized. However, they often involve a number of complicated structures and procedures that may not be familiar to most economists. In this first essay, we introduce another alternative to model fBm using a simple autoregressive approach. The key benefits of this approach are its simplicity, ease of implementation, fast computational speed, and robustness and consistency with the nature of fBm across the spectrum of Hurst exponents. First, we provide the proof to two key theorems, showing sufficient and necessary condition that ties the concept of unit root down to a simple mathematical relationship for an AR(q) process. We prove that any AR(q) process for which the sum of the lag coefficients equals one has a unit root, yet for any AR(q>1) this does not suggest an independent random walk process in increments. This finding provides clarity to a proper model specification for any AR process and will serve as a benchmark for the models developed in the latter parts of the dissertation. Then, fBm process and Hurst exponent are introduced formally and various proofs showing the relationship between the nature of memory AR process and fBm are presented. We also illustrate theoretical results using Monte Carlo simulation and a series of known fractional Brownian motions. A simple algorithm comprised of stepwise regression and restricted least squares is used to estimate the AR model's input parameters and we show that the resulting Hurst exponent from the simulation is remarkably consistent with the actual, known value. Finally, we extend our analysis to an unknown set of commodity futures price data and confirm the robustness and usefulness of the model and method.

In the second essay, we provide intuition and understanding to how an asset price path could behave across time space. To achieve that, we use Hurst exponent as a fractal gauge to capture the level of persistence in any given time series and exploit the dynamics of such variable using measures invoked from the Itô's excursion theory. To the best of our knowledge, we have not seen any studies that attempt to apply the Itô's excursion theory to fBm process and relate Hurst exponent to excursion measures. This essay is the first initiative of its kind. The key objectives of this essay can be summarized as follow.

Objective 1: To introduce various excursion measures from the Itô's excursion theory using simple example and basic mathematical concept.

Objective 2: To demonstrate how Itô's excursion theory can provide enormous clarity through graphical representation used to simplify mathematically involved stochastic processes.

Objective 3: To show that the excursions-valued process from the simulation actually follows a binomial distribution which is a robust substitute for Poisson distribution as suggested from the theory.

Objective 4: To illustrate the relationship between the Hurst exponent that characterized a certain stochastic process and the resulting excursion measures associated with such process.

In this second essay, we revisit the theory of excursion point process and apply the technique to detect and analyze the behavior of a known fractional Brownian motion. We demonstrate the key attributes and benefits of this tool via a step-by-step decomposition of the theory into simple mathematical terms. After that, we construct a program to measure excursion variables such as local time and excursion length based on the theory. Then, with a known set of data that behave as a pure Brownian motion, we capture this quality using the Itô's excursion theory and show that the excursions-valued process actually follows a binomial distribution which is a robust substitute for Poisson distribution as suggested from the theory. The ability to replicate the mathematical concept of the excursion theory using discrete-time Monte Carlo simulation then allow us to analyze the result observed when a known fractional Brownian motion is used as a data generator instead of the Brownian null. The excursion measures capture the characteristic differences among fractal processes with different underlying Hurst exponent. Specifically, a process with low Hurst exponent or short-memory process has higher mean excursion measure at low excursion length as compared to a process with high Hurst exponent or a long-memory process. On the other hand, we see systematic wandering with longer excursion in a long-memory process with Hurst exponent higher than 0.5. Finally, we conclude with an empirical application of the theory to an unrestricted set of commodity futures price data and again demonstrate the robustness of the method when applied to real market data.

In the third essay, we develop a trading strategy called Hurst Trading with trading signals generated from the fluctuation in the dynamics of the Hurst exponent across time, among other indicators. Again, to the best of our knowledge, this is the first study to come up with an explicit trading strategy involving Hurst exponent that attempts to make consistent profits across all fractal markets. The key objectives of this essay can be summarized as follow.

Objective 1: To enhance the performance of a momentum trading strategy by incorporating the information embedded in the Hurst exponent into the set of indicators used in generating the trading signals.

Objective 2: To introduce a rule-based algorithm that is the backbone to the Hurst Trading strategy.

Objective 3: To depict the profit and loss profile of the Hurst Trading strategy against the traditional momentum strategy and the Buy and Hold style based on the backtesting results on both synthetic and actual financial data.

Objective 4: To illustrate the relationship between the level of accuracy in timing the entry/exit points in the market of the Hurst Trading strategy and the degree of fractality in the time series data that is being exploited.

The concept of momentum trading is not new to us. It has been introduced and used by market practitioners for over a decade. However, similar to any proprietary trading strategies, the profit generated from momentum style of trading has diminished over the past decade as more and more people have adopted the strategy in their trading activities. One might argue that the huge influx of momentum arbitrageurs has eventually brought the market back to the efficient level, but we think it is time for new innovation. In this third essay, we extend the concept of traditional momentum trading to the fractal dimension by incorporating the use of the Hurst exponent estimation in order to come up with a more precise trading signal. This would allow one to better time the entry/exit points and thus be able to take advantage of both momentum and reversal in the markets. The ability of our model to predict reversal using the knowledge from the Hurst exponent parameter that characterized the time series illustrates how our approach is more superior and elegant than the traditional method. To achieve the goal, we first introduce a rule-based statistical arbitrage trading strategy, which we called Hurst Trading that integrates traditional momentum trading with Hurst exponent. As a point of comparison, we also present in this study a Buy and Hold style of trading. We employ a series of synthetic fractal processes with varying Hurst exponent as an input in our trading model. Then, we show the distribution of profits and losses generated from a simple market-neutral, Hurst Trading algorithm across time series with different Hurst exponent as well as several other interesting output variables. The results are strongly convincing that should the market deviate from being random as characterized by having a Hurst exponent equal to 0.5, consistent profit could be made by just following this simple trading strategy. To test the null of random and efficient market, we further apply the Hurst Trading strategy to real stock price data in various stock markets and again run a backtesting. We provide comparison in profits and losses generated from three types of trading strategies namely: Hurst Trading, traditional momentum trading, and Buy and Hold style. The Hurst Trading algorithm is shown to outperform the other two alternatives by a wide margin whenever the market is characterized by a Hurst exponent that is significantly different from 0.5, a fractal process. This is especially true among small-cap stocks in the Russell2500 Index.

CHAPTER II

AN AUTOREGRESSIVE APPROACH TO MODELING FRACTIONAL BROWNIAN MOTION

1. Introduction

In the past few decades, the development of several sophisticated derivatives markets and securities have sparked an outstanding number of research activities focusing on the pricing, modeling, and risk management of these products. A lot of attention has also been geared toward the study of time series analysis. Furthermore, the recent crashes, bubbles, and overall instability in the global financial markets have led to the search for the causes of these impactful events and the possible answer that would finally reveal what actually could have gone wrong in the system.

One, widely-criticized fact is that there is a huge mismatch between the model assumptions and the real world market. Markets are driven by players who do not necessarily act in a rational manner at all times. There are periods where psychological factors could dominate and drive the actions in the financial markets. This leads to several observed phenomena in the field of “Behavioral Finance” such as momentum trading, loss-aversion, and herding effect to name a few (Shiller 2000; Akerlof and Shiller 2009). Yet, most pricing models and risk management tools used by market practitioners are still relying heavily on a “nice”, random walk assumption for asset price behavior.

Nonetheless, financial economists have now started to realize that there are periods of jumps and turbulence in the market that could not be captured by the simple Brownian motion assumption. This naturally led researchers to start looking at fractional Brownian motion (fBm) as an alternative process to modeling asset prices. The development in this field of research is extraordinarily amazing, with the use of tools integrated from such diverse fields of study from physics to economics. In fact, the realization of the importance of interdisciplinary research actually brought about new fields of research such as “Econophysics” and “Behavioral Finance” (Stanley, Amaral et al. 1999; Barberis and Thaler 2003).

Different methods and techniques used to generate fBm have been documented in (Doukhan, Oppenheim et al. 2003). More recently, the generation of fBm using a wavelet-based approach has started to gain popularity due to its faster computational speed as compared to other simulation methods (Pipiras 2004). However, the implementation requires the generation of fractional ARIMA sequences with a suitable scaling parameter. The algorithm also demands a number of recursive steps involving wavelet transforms which may not be familiar to most economists.

In this paper, we introduce another alternative to model fBm using a simple autoregressive approach. The key benefits of this approach are its simplicity, ease of implementation, fast computational speed, and robustness and consistency with the nature of fBm across the spectrum of Hurst exponents. The paper is organized as follows. Section 2 provides the proof to two key theorems, showing sufficient and necessary condition that ties the concept of unit root down to a simple mathematical relationship for an AR(q) process. We prove that any AR(q) process for which the sum of the lag coefficients equals one has a unit root, yet for any AR(q>1) this does not suggest an independent random walk process in increments. This finding provides clarity to a proper model specification for any AR process and will serve as a benchmark for the models developed in the latter parts of the paper. In Section 3, fBm process and Hurst exponent are introduced formally and various proofs showing the relationship between the nature of memory AR process and fBm are presented. In Section 4, we illustrate theoretical results using Monte Carlo simulation and a series of known fractional Brownian motions. A simple algorithm comprised of stepwise regression and restricted least squares is used to estimate the AR model's input parameters and we show that the resulting Hurst exponent from the simulation is remarkably consistent with the actual, known value. In Section 5, we extend our analysis to an unknown set of commodity futures price data and confirm the robustness and usefulness of the model and method. We then conclude the paper.

2. Unit roots and autoregressive processes

An autoregressive (AR) process is a type of stochastic process that is often used to model and predict a vast array of economic phenomena including time-series models in financial economics (Greene and Zhang 2003). To understand the true meaning of the unit root in the context of stationary processes as it applies to financial price series, we offer the following proof.

Sufficient Condition $\sum_{i=1}^{q} a_i = 1$ ➔ existence of unit root

Theorem 1: Any sequence type AR(q), $Y_t = a_1 Y_{t-1} + a_2 Y_{t-2} + ... + a_q Y_{t-q} + \varepsilon_t$, in which the lag coefficients $\sum_{i=1}^{q} a_i = 1$ will have a unit root.

Proof: We can write $a_q = 1 - \sum_{i=1}^{q-1} a_i$. The characteristic polynomial for determining the real and complex roots of AR(q) is generally given by $v^q - \sum_{i=1}^{q-1} a_i v^{q-i} - a_q = 0$, which must be satisfied for any root *v*. By definition and substitution, we can write

$$v^q - \sum_{i=1}^{q-1} a_i v^{q-i} + \sum_{i=1}^{q-1} a_i - 1 = 0 .$$

Finally,

$$v^q - \sum_{i=1}^{q-1} a_i \left(v^{q-i} - 1 \right) - 1 = 0 .$$

Hence, without ambiguity, $v = 1$ is a solution and we have a unit root.

Q.E.D.

Necessary Condition existence of unit root ➔ $\sum_{i=1}^{q} a_i = 1$

Theorem 2: Any sequence type AR(q), $Y_t = a_1 Y_{t-1} + a_2 Y_{t-2} + ... + a_q Y_{t-q} + \varepsilon_t$, in which the lag coefficients $\sum_{i=1}^{q} a_i \neq 1$ will NOT have a unit root.

Proof: Now, let $\sum_{i=1}^{q} a_i \neq 1$. We can write $\sum_{i=1}^{q} a_i + R = 1$, where $R \neq 0$ is a remainder. The characteristic polynomial for determining the real and complex roots of AR(q) is generally given by $v^q - \sum_{i=1}^{q-1} a_i v^{q-i} - a_q = 0$, which must be satisfied for any root *v*. By definition and substitution, we can write

$$v^q - \sum_{i=1}^{q-1} a_i v^{q-i} + \sum_{i=1}^{q-1} a_i - 1 + R = 0.$$

Finally,

$$v^q - \sum_{i=1}^{q-1} a_i \left(v^{q-i} - 1 \right) - 1 + R = 0.$$

For v = 1 as a root, we end up with R = 0 which is a contradiction. Hence, there will be no unit root if $\sum_{i=1}^{q} a_i \neq 1$.

Q.E.D.

With the two theorems, we can conclude that $\sum_{i=1}^{q} a_i = 1 \leftrightarrow$ existence of unit root.

The theorems suggest and we will illustrate using the results obtained from Monte Carlo simulation that any sequence type AR(q), $Y_t = a_1 Y_{t-1} + a_2 Y_{t-2} + ... + a_q Y_{t-q} + \varepsilon_t$, in which the lag coefficients $\sum_{i=1}^{q} a_i \neq 1$ will have a limit in probability and time of either infinite or zero. Such a process does not exist for any meaningful economic applications in the long run. We will capture this quality using Hurst exponent and show the distribution of the fractional Hurst exponent around H = 0.5, the necessary and sufficient condition for a Brownian motion. With this, we have a better understanding of the coexistence between a random walk and temporary dependence among time increments. This point of view, which we have not been able to find within the financial economics literature, will be useful to the study of a wide range of applications, including price analysis, risk management, and hedging strategies.

3. Fractional Brownian motion under autoregressive process

Fractional Brownian motion is simply an extension of the well-known Brownian motion to the fractal dimensions. It was first introduced by Kolmogorov in 1940 when it was called *Wiener Helix*. Later, Mandelbrot and Van Ness gave the process its name *fractional Brownian motion* (Mandelbrot and Van Ness 1968; Campbell and Abhyankar 1978; Mandelbrot 1982). A fBm with Hurst exponent H belonging to (0,1) is a continuous and centered Gaussian process with covariance

$$E[B_t^H B_s^H] = \frac{\sigma^2}{2}\left(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\right) \qquad (s,t \geq 0),$$

where σ^2 is the variance of one-period return.

A fBm starts from zero almost surely, has stationary increments, and is self-affine[1]. For H = 0.5, the fBm becomes a standard Brownian motion where the increments are independent. If H < 0.5, the increments are negatively correlated resulting in a mean-reversion or ergodic process. When H > 0.5, they are positively correlated and lead to a

[1] A self-affine process, although similar in concept to a self-similar process, enjoys a higher degree of flexibility with respect to the scaling parameters in each dimension. The scaling factors do not necessarily have to be of the same magnitude across all dimensions. Generally speaking, self-similarity is a special case of self-affinity.

long-memory process (Bassingthwaighte and Raymond 1994; Carmona and Coutin 1998; Alvarez-Ramirez, Cisneros et al. 2002; Turvey 2007; Biagini, Hu et al. 2008).

The standard fBm $B^{(H)}$ has the following properties:

1. $B^{(H)}(0) = 0$ and $E\,[B^{(H)}(t)] = 0$ for all $t \geq 0$.
2. $B^{(H)}$ has stationary increments.
3. $B^{(H)}$ has continuous trajectories.
4. $B^{(H)}$ is a Gaussian process.

It is quite straightforward to show that under certain conditions we can construct an AR process that has the first 3 properties above, but it is not clear from the beginning whether the constructed process is actually Gaussian. Hence, we provide a short proof as sketched below.

Theorem 3: An autoregressive process AR(q) with random innovation ε that is normally distributed or Gaussian will also be a Gaussian process.

Proof: A random process Y_t is Gaussian if the joint distribution of each element $Y_t, Y_{t+1}, Y_{t+2}, \ldots$ follows a normal distribution. A linear transformation of Gaussian processes yield another Gaussian process.

For AR(1), we have $Y_t = a_0 + a_1 Y_{t-1} + \varepsilon_t$

$$= a_0 + a_1(a_0 + a_1 Y_{t-2} + \varepsilon_{t-1}) + \varepsilon_t$$

$$= a_0(1 + a_1) + a_1^2 Y_{t-2} + a_1 \varepsilon_{t-1} + \varepsilon_t$$

.

. repeat N times

.

$$= a_0 \sum_{i=0}^{N} a_1^i + \sum_{i=0}^{N} a_1^i\, \varepsilon_{t-i} + a_1^{N+1} Y_{t-(N+1)}$$

.

. WLOG, assume $Y_0 = 0$ as scaling is feasible

.

As N→∞, we have $Y_t = \frac{a_0}{1-a_1} + \sum_{i=0}^{\infty} a_1^i \varepsilon_{t-i}$

If $\varepsilon_t \sim N(0, \sigma^2)$, then ε_t is a Gaussian process and Y_t is also Gaussian since it can be written as a linear combination of ε.

We can extend this procedure to any AR(q) and in the limit as N→∞ we will again end up with an expression $Y_t = c + f(\varepsilon; a)$, where c = constant and $f(\varepsilon; a)$ is a linear function of random innovation terms and lag coefficients. Hence, if ε_t is a Gaussian process, Y_t will also be Gaussian.

Q.E.D.

Since our interest is on price data, we would like to implement an autoregressive model that restricts price from falling below the zero level. In this paper, we use the following specific multivariate polynomials form

$$X_t = e^u X_{t-1}^{a_1} X_{t-2}^{a_2} \cdots X_{t-q}^{a_q} e^{\varepsilon_t} , \tag{1}$$

with a constant drift u and a random innovation ε that is normally distributed with mean zero and variance σ^2. This allows our price level to follow a lognormal distribution. Taking natural log transformation on both side of Eq. (1), we then have a log-linear model

$$Y_t = u + a_1 Y_{t-1} + a_2 Y_{t-2} + ... + a_q Y_{t-q} + \varepsilon_t , \tag{2}$$

where Y_t is the log price at time t.

3.1 Hurst exponent as a fractal gauge

It was Mandelbrot who coined the parameter H Hurst exponent after the name of a British hydrologist Harold Edwin Hurst, who studied the yearly water run-offs in the Nile River basin (Biagini, Hu et al. 2008). In his study, Hurst discovered that the values of successive yearly run-offs show a certain level of dependency. This phenomenon could not be modeled using a process with independent increments so he developed a method that eventually became known today as the Hurst rescaled range analysis (*R/S*). Each successive run-off could be thought of as the increment of a fBm characterized by a certain value of the Hurst exponent (Hurst 1951; Bassingthwaighte and Raymond 1994).

In this paper, we will use the scaled variance ratio technique from (Cannon, Percival et al. 1997; Turvey 2007) that is quite distinct but consistent with *R/S* analyses (Hurst 1951; Mandelbrot and Van Ness 1968) to estimate the Hurst exponent of a fBm generated from an AR(q) process:

$$\frac{E\left[Y(t+k)-Y(t)\right]^2}{E\left[Y(t+1)-Y(t)\right]^2}=\frac{\sigma_k^2}{\sigma_1^2}=\left(k\right)^{2H}, \tag{3}$$

which defines a power rule that can be used to estimate the value of *H*.

3.2 Relationship between variance ratio and AR lag coefficients

One of the key benefits we obtain from generating a fBm using an AR process is that we can disentangle the concept of stationarity from independence in increments. This is extremely important as it articulates the fine line separating a pure random walk from a memory process. We offer a mathematical derivation below that demonstrates how the choice of lag coefficients in the AR model could affect the variance ratio of the resulting process.

Lemma 1: If the process exhibits stationary increments, then it has a unit root.

Proof: For any AR(q) process, we have

$$Y_t = a_0 + a_1 Y_{t-1} + a_2 Y_{t-2} + \cdots + a_q Y_{t-q} + \varepsilon_t$$

$$Y_t - Y_{t-1} = a_1(Y_{t-1} - Y_{t-2}) + a_2(Y_{t-2} - Y_{t-3}) + \cdots + a_q\left(Y_{t-q} - Y_{t-q-1}\right) + \varepsilon_t - \varepsilon_{t-1}$$

Taking expectation on both sides,

$$E[Y_t - Y_{t-1}] = a_1 E[Y_{t-1} - Y_{t-2}] + a_2 E[Y_{t-2} - Y_{t-3}] + \cdots + a_q E\left[Y_{t-q} - Y_{t-q-1}\right]$$

Stationary increments implies

$$E[Y_t - Y_{t-1}] = E[Y_{t-1} - Y_{t-2}] = \cdots = E\left[Y_{t-q} - Y_{t-q-1}\right]$$

Therefore, we have

$$E[Y_t - Y_{t-1}] = (a_1 + a_2 + \cdots + a_q)E[Y_t - Y_{t-1}]$$

$$\sum_{i=1}^{q} a_i = 1$$

From Theorem 1 and 2, this process has a unit root.

Q.E.D.

The variance of the one time-step increment can be expressed as

$$Var[Y_t - Y_{t-1}] = Var[Z_1] = E[Z_1^2] - E[Z_1]^2$$

where $$Z_1 = Y_t - Y_{t-1}$$

$$= a_1(Y_{t-1} - Y_{t-2}) + \cdots + a_q\left(Y_{t-q} - Y_{t-q-1}\right) + \varepsilon_t - \varepsilon_{t-1}$$

$$= a_1 Z_2 + \cdots + a_q Z_{q+1} + \varepsilon_t - \varepsilon_{t-1}$$

$$Z_i = Y_{t-(i)+1} - Y_{t-(i)}$$

$$Z_{i+1} = Y_{t-i} - Y_{t-i-1}$$

Hence, we have

$$Var[Z_1] = E\left[\sum_{i=1}^{q} a_i^2 Z_{i+1}^2 + 2\sum_{i=1,j=1,i<j}^{q} a_i a_j Z_{i+1} Z_{j+1} + \varepsilon_t^2 + \varepsilon_{t-1}^2 - 2\varepsilon_t\,\varepsilon_{t-1}\right.$$

$$\left. + 2\varepsilon_t \sum_{i=1}^{q} a_i Z_{i+1} - 2\varepsilon_{t-1}\sum_{i=1}^{q} a_i Z_{i+1}\right] - (a_1 E[Z_2] + \cdots + a_q E\left[Z_{q+1}\right])^2$$

$$= E\left[\sum_{i=1}^{q} a_i^2 Z_{i+1}^2 + 2\sum_{i=1,j=1,i<j}^{q} a_i a_j Z_{i+1} Z_{j+1} - 2\varepsilon_t\,\varepsilon_{t-1}\right] + 2\sigma^2 - \sum_{i=1}^{q} a_i^2 E[Z_{i+1}]^2$$

$$- 2\sum_{i=1,j=1,i<j}^{q} a_i a_j E[Z_{i+1}] E[Z_{j+1}]$$

$$= \sum_{i=1}^{q} a_i^2 (E[Z_{i+1}^2] - E[Z_{i+1}]^2) + 2\sum_{i=1,j=1,i<j}^{q} a_i a_j (E\left[Z_{i+1} Z_{j+1}\right] - E[Z_{i+1}] E\left[Z_{j+1}\right])$$

$$+ 2\sigma^2 - 2Cov[\varepsilon_t, \varepsilon_{t-1}]$$

(4)

Lemma 2: $Var[Y_t - Y_{t-k}] = kVar[Y_t - Y_{t-1}] + 2\sum_{i=1,j=1,i<j}^{k} Cov\left[Z_i, Z_j\right]$

Proof:

$$Y_t - Y_{t-2} = Y_t - Y_{t-2} + (Y_{t-1} - Y_{t-1})$$

$$= (Y_t - Y_{t-1}) + (Y_{t-1} - Y_{t-2})$$

$$Y_t - Y_{t-k} = Y_t - Y_{t-k} + (Y_{t-1} - Y_{t-1}) + (Y_{t-2} - Y_{t-2}) + \cdots + (Y_{t-k+1} - Y_{t-k+1})$$

$$= (Y_t - Y_{t-1}) + (Y_{t-1} - Y_{t-2}) + (Y_{t-2} - Y_{t-3}) + \cdots + (Y_{t-k+1} - Y_{t-k})$$

With stationary but not necessarily independent increments assumption, we have

$$Var[Y_t - Y_{t-k}] = kVar[Y_t - Y_{t-1}] + 2\sum_{i=1,j=1,i<j}^{k} Cov\left[Z_i, Z_j\right] \qquad (5)$$

Q.E.D.

Finally, from Eq. (5) we can derive the variance ratio formula as follow

$$\frac{Var[Y_t - Y_{t-k}]}{Var[Y_t - Y_{t-1}]} = k + \frac{2\sum_{i=1,j=1,i<j}^{k} Cov\left[Z_i, Z_j\right]}{Var[Y_t - Y_{t-1}]}$$

(6)

We can further substitute the expression for the one-period variance from Eq. (4) into Eq. (6) to get the final equation that defines the relationship between variance ratio and AR lag coefficients.

$$\frac{Var[Y_t-Y_{t-k}]}{Var[Y_t-Y_{t-1}]}=k+\frac{2\sum_{i=1,j=1,i<j}^{k}Cov[Z_i,Z_j]}{\sum_{i=1}^{q}a_i^2(E[Z_{i+1}^2]-E[Z_{i+1}]^2)+2\sum_{i=1,j=1,i<j}^{q}a_ia_j(E[Z_{i+1}Z_{j+1}]-E[Z_{i+1}]E[Z_{j+1}])+2\sigma^2-2Cov[\varepsilon_t,\varepsilon_{t-1}]} \quad (7)$$

In Eq. (7), the left hand side of the equation is the variance ratio which was described previously in Eq. (3). The first term on the right hand side of the equation is the lag length *k* at which the variance ratio is being evaluated. The second term constitutes the key element that defines the fractal properties of an AR process. The numerator represents the covariance structure of the process. The denominator is comprised of three parts: the first is due to stationarity, the second term scales for the variance, and the third part adjusts for autocorrelation effect.

Moreover, it is worthwhile to note that we can actually write an equation that relates the AR lag coefficients to the Hurst exponent that describes the characteristic of the process. Combining Eq. (3) and (7) and solve for *H*, we obtain

$$H = \frac{\ln\left(k + \frac{2\sum_{i=1,j=1,i<j}^{k} Cov[Z_i, Z_j]}{\sum_{i=1}^{q} a_i^2 (E[Z_{i+1}^2] - E[Z_{i+1}]^2) + 2\sum_{i=1,j=1,i<j}^{q} a_i a_j (E[Z_{i+1} Z_{j+1}] - E[Z_{i+1}] E[Z_{j+1}]) + 2\sigma^2 - 2Cov[\varepsilon_t, \varepsilon_{t-1}]}\right)}{2\ln(k)} \tag{8}$$

It is then straightforward to see that in the absence of memory as reflected through the covariance structure being zero, Eq. (8) collapses to give a value of H = 0.5, the case of a pure random walk.

3.3 Simulating a fBm using an AR process

In this part of the paper, we will utilize the theorems shown in Section 2 to generate fBm from AR(q>1) with $\sum_{i=1}^{q} a_i = 1$ and an innovation term that is normally distributed. According to the theorems, this is a legitimate process with a unit root but we will embed in it some dependency in increments through the higher-order autoregressive terms.

First, we show that any sequence type AR(q) given in Eq. (2), in which the lag coefficients $\sum_{i=1}^{q} a_i \neq 1$, will have a limit in probability and time of either infinite or zero. This reminds us that in order to capture any meaningful economic phenomena in the long run, we should be careful about the way we specify the lag coefficients in our autoregressive model. In fact, the limit will tend toward infinity when $\sum_{i=1}^{q} a_i > 1$ and toward zero when $0 < \sum_{i=1}^{q} a_i < 1$.

However, the process could still be badly behaved even though we have $\sum_{i=1}^{q} a_i = 1$. For example, we could have a price process with $\sum_{i=1}^{q} a_i = 1$, a unit-root process, that either explodes or goes to zero in the limit depending on the values of the lag coefficients. Therefore, in order to ensure long-run stability, we need to impose more restriction on the lag coefficients. One possible set of restrictions might be to have $0 \le a_i \le 1$ and $\sum_{i=1}^{q} a_i = 1$ for all *i*. When estimating an AR(q) model under the null $\sum_{i=1}^{q} a_i = 1$, the coefficients and predictive model will likely be stable. But from the data generation perspective, the choice of lag coefficients can lead to extraordinary and seemingly meaningless processes.

Figure 1 shows some results from a Monte Carlo simulation run of 10,000 iterations. We use sample size of $N = 2520$, which is approximately equal to 10 years of daily price data. Without loss of generality, we apply innovation term that is normally distributed with zero mean and standard deviation $\sigma = 0.25$. The three cases shown in Figure 1 indicate the limiting nature of the AR process based on its coefficients parameter. The series will explode in the limit when $\sum_{i=1}^{q} a_i > 1$ and will drift toward zero when $0 < \sum_{i=1}^{q} a_i < 1$. The only stable case in the long run is when $\sum_{i=1}^{q} a_i = 1$.

Figure 1: Limiting behavior for three different types of AR processes

Therefore, in order to have an autoregressive model that is appropriate for any long-run economic phenomena (i.e., not exploding or reaching a very low value in the limit), the first screening test on the model specification is to have $\sum_{i=1}^{q} a_i = 1$. Then, we also need to ensure that the estimated lag coefficients would not lead to an unstable autoregressive process. In other words, it ought to be a unit-root autoregressive process with proper specification on the lag coefficients.

3.4 Estimating the Hurst exponent from simulation

Again, the methods involve first the generation of a 2520-day path of price levels using Eq. (2). Without loss of generality, we use a starting price of 100, zero drift, and an innovation's standard deviation of 0.25. Also, with the structure of the AR(q) program, we have the freedom to analyze different combinations of lag coefficients in the process.

Next, we calculate the natural logarithm of the price levels to come up with the log-return or $Y_{t+k} - Y_t$, for each nonnegative t and allowing for overlapping prices[2] ($k = 1,2,\ldots,50$).

Third, we calculate the variance, $Var(Y_{t+k} - Y_t)$, for each value of k.

Fourth, we divide the calculated variance for each $k \geq 1$ by the variance for $k = 1$ to form the variance ratio $\frac{\sigma_k^2}{\sigma_1^2}$.

Fifth, we estimate the value of H from Eq. (3) which leads to

$$H = \frac{\ln\left(\frac{\sigma_k^2}{\sigma_1^2}\right)}{2\ln(k)}. \tag{9}$$

Table 1 shows some results from a Monte Carlo simulation run of 10,000 iterations. Different combinations of lag coefficients are exploited to confirm our hypothesis. We

---

[2] Recent literatures (Lo and Mackinlay 2001; Ellis 2007) have pointed out that overlapping and contiguous subseries are equally valuable when the sample size concerned is large. However, overlapping subseries become distinctly superior when dealing with limited data range.

can observe that the estimation result for the mean Hurst exponent is statistically within the proximity of 0.5 for all values of *k* for AR(1) process with $\sum_{i=1}^{q} a_i = 1$, the case of a random walk.

Of particular interest is the result for the case of AR(q>1) with $\sum_{i=1}^{q} a_i = 1$. The theorems we proposed in Section 2 confirm the validity of the existence of a unit root in this process. However, despite having stationary increments, the increments are not independent for this specific cases since we have AR(q>1) process where memory is embedded within the process generator. The estimation of the mean Hurst exponent in row 4, 5, 7, and 8 supports this argument and shows Hurst exponents that are statistically different from 0.5. Thus, a fBm process is a unit-root process but contain a degree of persistency in time. Consequently, any tests that reject the null of a unit root in such process suffer Type I error, a situation whereby the null hypothesis is rejected when it is in fact true.

Table 1: This table shows the mean Hurst exponent from simulation evaluated at lag k = 10 to 50 for different AR(q) structures with q = 1 to 3

q	**a1**	**a2**	**a3**	**Lag**	**10**	**20**	**30**	**40**	**50**
1	1	0	0	**Mean H**	0.4987	0.4981	0.4974	0.4968	0.4961
1	0.99	0	0	**Mean H**	0.4910	0.4852	0.4800	0.4753	0.4708
1	0.5	0	0	**Mean H**	0.1569	0.1210	0.1065	0.0984	0.0925
2	0.8	0.2	0	**Mean H**	0.4191	0.4333	0.4394	0.4429	0.4451
2	0.2	0.8	0	**Mean H**	0.0804	0.1540	0.1853	0.2044	0.2178
2	0.5	0.4	0	**Mean H**	0.2608	0.2614	0.2508	0.2399	0.2297
3	1.2	-0.5	0.3	**Mean H**	0.4496	0.4528	0.4553	0.4568	0.4578
3	0.2	0.3	0.5	**Mean H**	0.1010	0.1538	0.1816	0.1993	0.2119
3	0.6	0.1	0.2	**Mean H**	0.2630	0.2628	0.2531	0.2426	0.2329

3.5 Comparison between estimates from analytical formula and simulation

At this point, we have an AR model that can generate any realization of price path based on any given set of input parameters. The algorithm also calculates the one-period variance, covariance matrix, variance ratio, and Hurst exponent for all lags from k = 2 to 50. At the same time, we have derived a series of analytical formulas that define the theoretical value of all these variables. This makes it convenient and natural to calibrate the model by comparing the model outputs against their theoretical values.

For the purpose of illustration, a sample case with k = 5 and q = 2 is chosen. The total observation is again 2520 data points. The starting price is set to 100 with the innovation's standard deviation equal to 0.3. The coefficients in the AR(2) model are constrained so that the sum of the coefficients equals unity with one coefficient being randomly generated from a Uniform(0.1,2) distribution.

$$Y_t = a_1 Y_{t-1} + (1 - a_1) Y_{t-2} + \varepsilon_t$$

where $\varepsilon_t \sim N(0,0.3)$

$a_1 \sim U(0.1,2)$

Monte Carlo simulation is run again for 10,000 iterations. Figure 2 shows the distribution of the average Hurst exponent from k = 2 to 50 around the mean of 0.504.

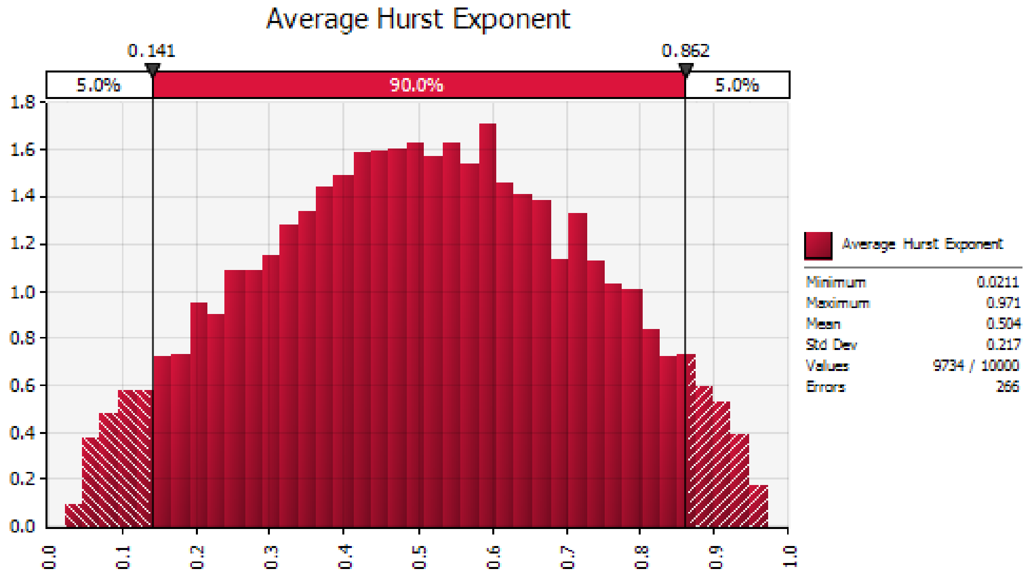


Figure 2: Distribution of average Hurst exponent from simulation

The comparison between the model outputs and their theoretical values are presented in Table 2 in the form of a ratio. The ideal case would be to have a ratio that equal to one for all four variables concerned. This would represent the situation where the result from Monte Carlo simulation matches perfectly with the number obtained from the analytical formula. The results for all four variables are solid with means that are very close to one. Therefore, any estimation obtained from the Monte Carlo simulation using AR model is considered robust and consistent with the theory.

Table 2: Model calibration statistics

Analytical/Simulated Ratio	Mean	Std Dev	Min	Max
One-Period Variance	0.9979	0.0038	0.9698	1.0004
Covariance	0.9687	0.1248	0.5925	1.2741
Variance Ratio	0.9900	0.0148	0.8700	1.0078
Hurst Exponent	0.9633	0.1310	-0.7802	1.0402

Moreover, a few interesting relationships are found and presented in the form of scatter plots between each pair of variables. First, the average Hurst exponent around 0.5 corresponds to the case where the analytical one-period variance equal to σ^2 as shown in Figure 3. This should be quite intuitive as an AR(1) process with lag coefficient equal to one typically yields a Hurst exponent estimation around 0.5, and we know that the variance of the log-return for such process is simply equal to the variance of the white noise itself. As the process becomes more fractal, the variance increases following the relationship defined in Eq. (4). Second, the covariance values obtained from simulation track their theoretical values very closely except for some processes with very low Hurst exponent where the simulated number tends to overshoot the value suggested by the analytical formula. However, the result shows that the ratio does fall within the 95% confidence interval [0.724, 1.214] in most cases. This is shown as a scatter plot in Figure 4.

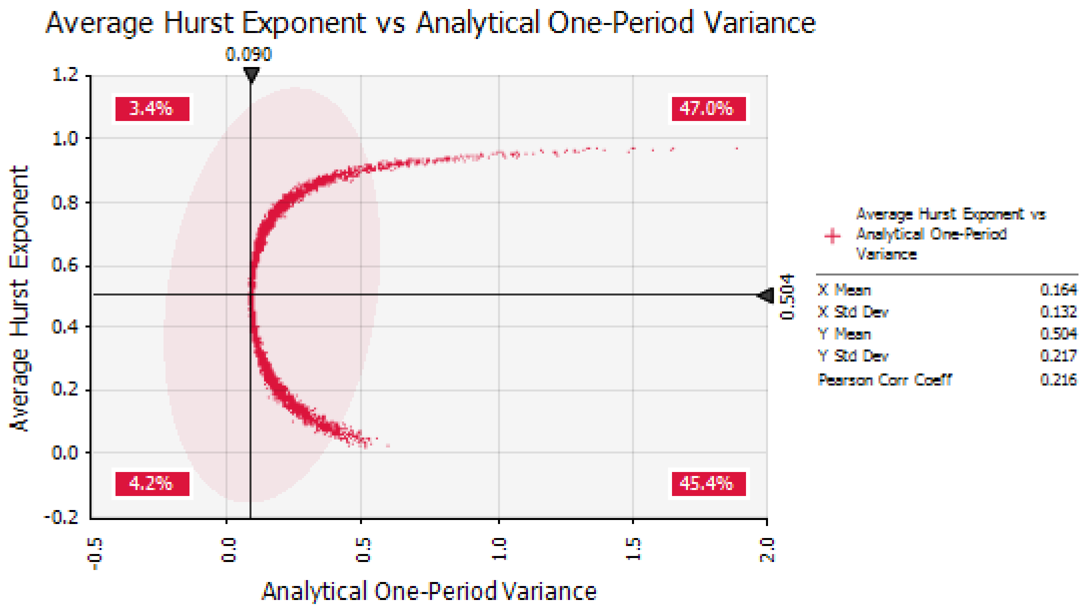


Figure 3: Relationship between Hurst exponent and variance

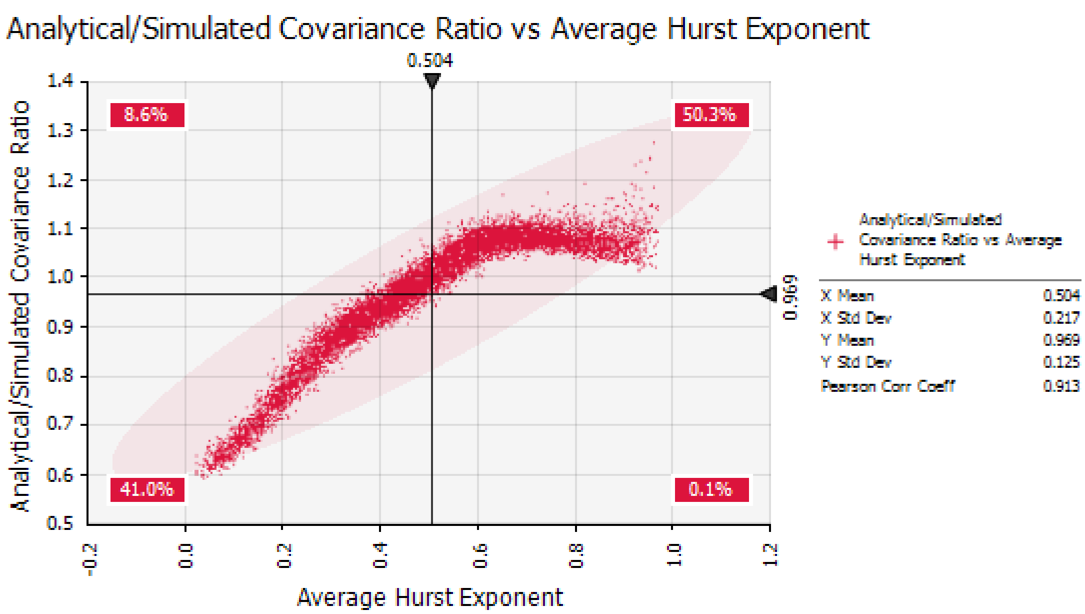


Figure 4: Relationship between Hurst exponent and A/S covariance ratio

4. Empirical results on a known fBm series

After demonstrating the consistency of the AR model against the theory through a series of proofs and calibrations with analytical formula, the next extension is the model implementation part through a series of empirical tests. The first test will be on a known fBm series.

4.1 Data description

In this section, a known fBm series with Hurst exponent ranging from 0.1 to 0.9 are generated in Matlab using the algorithm proposed by (Abry 1996). Each series contains 2150 data points. Basically, the general idea of the algorithm is to build a biorthogonal wavelet that depends on a given orthogonal wavelet and adapted to a specific Hurst exponent parameter H. The fBm process can be expressed as a fractional integral of the white noise process. Thus, the generated sample path is obtained by the reconstruction using the new wavelet initiated from the wavelet decomposition technique applied to a certain level of interest. A comprehensive overview of the implementation technique can be found in (Abry 1996; Bardet, Lang et al. 2003). The fBm series is shown in Figure 5 below.

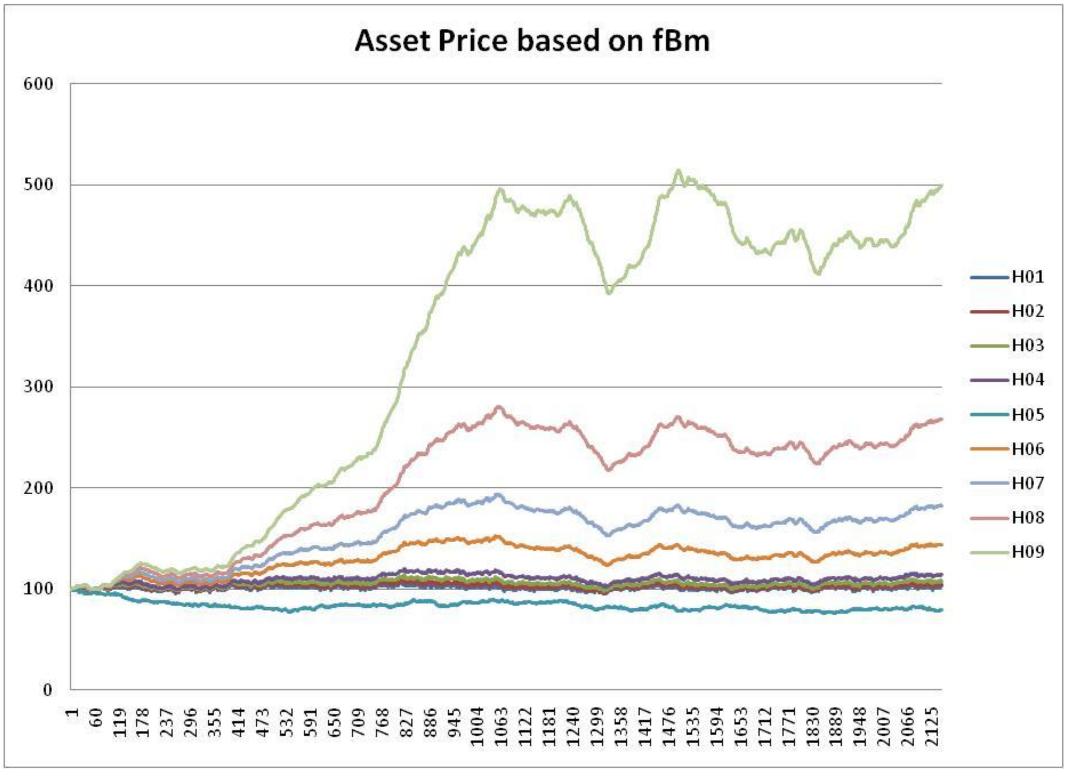


Figure 5: Fractional Brownian motion time series

4.2 Method and result

A preliminary test is first carried out by estimating the Hurst exponent of these known fBm data set using our AR model. The results are shown in Table 3 under "Actual H" for each time series. The Hurst exponent estimators match the known Hurst input parameters very closely for all series except for those at the two extreme tails, H01 and H09, where there is a slight difference. In general, the results reveal a reasonably high level of consistency across the entire spectrum of Hurst exponent.

The next step is to introduce an algorithm to replicate the known fBm series using AR model such that the new time series will be consistent in nature to the original series based on their specific Hurst exponent value. In order to achieve this goal, each known fBm series is used to generate a set of lag series up to 50 lags. This data set serves as the input in STATA where a stepwise regression is performed to find the AR lag structure that provides the best fit to the fBm series. Additionally, a test of linear restriction is implemented to ensure that any lag structure recommended actually satisfies the necessary and sufficient condition of a unit root process with the sum of the estimated coefficients equals unity. This is shown to be true at 5% significant level for the whole set of known fBm series used.

For example, the AR structures found for H03 and H07 series are as followed.

<u>H03:</u> $Y_t = 0.7120234Y_{t-1} + 0.1362426Y_{t-2} + 0.0731965Y_{t-3} + 0.0406646Y_{t-6} + 0.0378728Y_{t-26} + \varepsilon_t$

<u>H07:</u>

$Y_t = 1.239981Y_{t-1} - 0.1630198Y_{t-2} - 0.03901Y_{t-3} - 0.0334447Y_{t-8} - 0.0045068Y_{t-50} + \varepsilon_t$

The complete regression statistics and test results can be found in Appendix A.

Finally, with the AR coefficient estimators from the regression, Monte Carlo simulation is run for 10,000 iterations on the AR model with initial value being set equal to the mean of the original series and standard deviation of the random shock following the expression, $\sigma = (Root\ MSE)(\sqrt{Number\ of\ observation})$. The mean Hurst exponent at each lag is recorded and shown in Table 3 under "Simulated Mean H".

As suggested by the study in (Weron 2002), the recommended lag (k) to analyze the Hurst estimator follows the relationship defined by $k = \sqrt{N}$, where N = 2150 in our study. Hence, we might want to pay particular attention to comparing the "Actual H" against the "Simulated Mean H" at lag k = 46. Table 3 shows that the results match very precisely to two decimal points for all but the series H09 which contains lots of noise[3].

Table 3: Actual Hurst exponent versus simulated result

Lag			5	10	20	30	35	46	50
H01	Actual	H	0.15	0.15	0.15	0.14	0.14	0.14	0.14
	Simulated	Mean H	0.16	0.16	0.16	0.14	0.14	0.14	0.15
H02	Actual	H	0.22	0.23	0.23	0.22	0.22	0.22	0.22
	Simulated	Mean H	0.23	0.24	0.23	0.22	0.22	0.22	0.23
H03	Actual	H	0.30	0.32	0.32	0.31	0.30	0.31	0.31
	Simulated	Mean H	0.31	0.32	0.31	0.30	0.30	0.30	0.30
H04	Actual	H	0.40	0.41	0.41	0.40	0.40	0.40	0.41
	Simulated	Mean H	0.40	0.41	0.40	0.39	0.39	0.40	0.40
H05	Actual	H	0.50	0.51	0.50	0.50	0.50	0.50	0.50
	Simulated	Mean H	0.50	0.50	0.50	0.50	0.50	0.50	0.49
H06	Actual	H	0.60	0.61	0.60	0.60	0.59	0.60	0.60
	Simulated	Mean H	0.60	0.61	0.60	0.59	0.59	0.59	0.58
H07	Actual	H	0.70	0.70	0.70	0.69	0.69	0.69	0.69
	Simulated	Mean H	0.72	0.72	0.72	0.72	0.71	0.71	0.71
H08	Actual	H	0.79	0.79	0.78	0.78	0.78	0.78	0.78
	Simulated	Mean H	0.86	0.85	0.84	0.83	0.83	0.83	0.83
H09	Actual	H	0.87	0.87	0.86	0.86	0.85	0.85	0.85
	Simulated	Mean H	0.97	0.97	0.95	0.94	0.94	0.94	0.94

[3] We have not seen a single study with which a Hurst exponent as high as H = 0.9 has been found for a time series of any financial assets.

5. Empirical results on commodity futures price data

In order to demonstrate the practical usefulness of this simple AR approach to modeling fBm, a similar empirical test is conducted on commodity futures price data obtained from traded commodity futures contracts listed on the Chicago Mercantile Exchange. Hence, in contrast to the known fBm data set used in Section 4, the defining Hurst characteristic of each commodity futures time series is not known beforehand. And this is considered extremely relevant and useful to modeling any real world applications since most financial and physical phenomena do not behave like a pure random walk at all time period. The ability of the AR model to replicate any unknown time series and preserve their Hurst characteristic is extremely important and useful to practitioners who wish to have a model with minimum assumptions that is able to capture the risk attributes effectively.

5.1 Data description

Time-series futures price data of five commodities are used in this study. They are live cattle, wheat, lean hogs, pork bellies, and coffee. The data represents 952 daily observations from 03/21/96 to 02/09/00 for each commodity.

Figure 6: Commodity futures prices

5.2 Result

The regression and estimation results confirm our earlier hypothesis about the nature of the unit root process and the embedded dependency among increments through the use of an autoregressive process generator. The estimated AR equations for all commodities are as followed.

<u>Live Cattle:</u>

$$Y_t = 0.9834966Y_{t-1} - 0.066398Y_{t-4} + 0.069461Y_{t-7} - 0.01256954Y_{t-48} + 0.1391358Y_{t-49} + \varepsilon_t$$

Wheat:

$$Y_t = 0.9291032Y_{t-1} + 0.1237614Y_{t-3} - 0.1307927Y_{t-4} + 0.1015567Y_{t-6} - 0.0536245Y_{t-9} + 0.0221065Y_{t-28} - 0.2296391Y_{t-48} + 0.2375285Y_{t-49} + \varepsilon_t$$

Lean Hogs:

$$Y_t = Y_{t-1} + \varepsilon_t$$

Pork Bellies:

$$Y_t = 1.025506Y_{t-1} - 0.025506Y_{t-7} + \varepsilon_t$$

Coffee:

$$Y_t = 0.986107Y_{t-1} - 0.0522767Y_{t-29} + 0.1251542Y_{t-31} - 0.0589844Y_{t-33} + \varepsilon_t$$

A full set of regression statistics and test results can be found in Appendix B.

The resulting Hurst exponent estimations are based on a lag structure with k = 31 days as recommended by (Weron 2002) that $k = \sqrt{N}$, where N = 952 in this data set. However, it is shown that the null that demands the sum of the estimated coefficients from the regression to be equal to unity is rejected at 5% significant level in the test of linear restriction for wheat. The test shows a p-value of 0.0099 which leads to a rejection of the null. This result has led us to realize one limitation of the model which is the sample size issue. In this part of the study, the sample size is only 952 data points. As such, it is more likely that there will be rejection of the null hypothesis in the F-test of linear restriction since the degree of freedom will be reduced substantially if the lag structure requires the use of several lag variables. This is exactly what is happening in the case of wheat since the stepwise regression shows significant coefficient estimators for 8 different lag variables. This also implies that the wheat time series exhibits stronger fractal characteristics. However, the scope of the problem is only limited to the consistency of the model to a unit root process and this could be easily fixed by increasing the sample size. It is also worth mentioning that most of the time it is not difficult to obtain historical data up to a few years especially for financial time series so this problem could be conveniently mitigated in most applications.

With this algorithm, we are able to mimic the characteristic of the original time series based on the Hurst exponent estimation. Table 4 shows the comparison of the "Actual H" against "Simulated Mean H" values for all five commodities. Three out of five commodities, namely wheat, pork bellies, and coffee, have these two estimations exactly match up to two decimal points at lag k = 31. This again reveals the robustness of our estimation algorithm. Within this data set, wheat contracts seem to exhibit some mean-reversion characteristics with Actual H = 0.40. Overall, the other four commodities tend to behave in a consistent manner to a pure random walk process. This implies that the specific futures market is somewhat efficient and would be difficult to arbitrage solely on the knowledge of dependency in price returns.

Table 4: Actual Hurst exponent versus simulated result

Lag			5	10	15	20	31	40	50
Live Cattle	Actual	H	0.47	0.42	0.41	0.40	0.43	0.43	0.41
	Simulated	Mean H	0.49	0.44	0.43	0.42	0.41	0.41	0.39
Wheat	Actual	H	0.45	0.44	0.43	0.41	0.40	0.39	0.38
	Simulated	Mean H	0.45	0.45	0.44	0.43	0.40	0.39	0.39
Lean Hogs	Actual	H	0.48	0.50	0.51	0.51	0.52	0.51	0.50
	Simulated	Mean H	0.50	0.50	0.50	0.49	0.49	0.49	0.49
Pork Bellies	Actual	H	0.53	0.53	0.53	0.53	0.53	0.51	0.50
	Simulated	Mean H	0.53	0.54	0.54	0.54	0.53	0.53	0.53
Coffee	Actual	H	0.49	0.47	0.45	0.45	0.44	0.43	0.44
	Simulated	Mean H	0.48	0.48	0.47	0.46	0.44	0.43	0.43

6. Conclusion

It is undeniable that fBm process would start to play an increasingly important role in modeling the behavior of certain asset markets in the field of economics study due to its flexibility and ability to capture short and long range dependency in price return. This study outlines a simple, yet robust and consistent method to generate fBm using an autoregressive approach that is familiar to economists. It is shown that having the sum of the lag coefficients equal to unity in an AR(q) process is a necessary and sufficient condition for the existence of a unit root. Furthermore, any sequence type AR(q) with lag coefficients that do not sum up to unity will have a limit in probability and time of either infinite or zero.

An equation that defines the relationship between the AR lag coefficients and the Hurst exponent that described a particular fBm process is derived. This finding is crucially important since it allows one to understand how a choice of lag coefficients could affect the resulting behavior of the process generated from the model. It also allows one to solve for an unknown variable in the defining equation given the knowledge of the rest of the parameters. This gives researchers full control over the particular process they wish to generate.

It is also shown in the study that this method of modeling fBm is generally robust across the spectrum of Hurst exponents except for those processes at the two extreme tails. However, this limitation should not affect the practical usefulness of using this method to model any fBm processes in the context of agricultural or financial markets since most markets tend to exhibit memory within the range of H = [0.3,0.7] only (Opong, Mulholland et al. 1999; Cajueiro and Tabak 2004; Turvey 2007; Eom, Choi et al. 2008). It is very rare or almost non-existent to find a meaningful process in the real world with Hurst exponent outside this range.

Finally, the findings in this study have some important implications to industry practitioners who require a simulation method that is fast and easy to implement, yet consistent with the theory. Meaningful applications could be extended to market risk management, pricing model, program trading, etc. Practitioners can now better analyze and distinguish between short term dependency and long term randomness given this understanding.

REFERENCES

Abry, P. (1996). The wavelet-based synthesis for fractional Brownian motion proposed by F. Sellan and Y. Meyer: Remarks and fast implementation, Applied and Computational Harmonic Analysis. **3:** 377-383.

Akerlof, G. A. and R. J. Shiller (2009). Animal spirits: How human psychology drives the economy, and why it matters for global capitalism, Princeton University Press.

Alvarez-Ramirez, J., M. Cisneros, et al. (2002). Multifractal Hurst analysis of crude oil prices, Physica A: Statistical Mechanics and its Applications. **313:** 651-670.

Barberis, N. and R. Thaler (2003). A survey of behavioral finance, Handbook of the Economics of Finance. **1:** 1053-1128.

Bardet, J. M., G. Lang, et al. (2003). Generators of long-range dependent processes: a survey, Theory and Applications of Long-Range Dependence. 579-623.

Bassingthwaighte, J. B. and G. M. Raymond (1994). Evaluating rescaled range analysis for time series, Annals of Biomedical Engineering. **22:** 432-444.

Biagini, F., Y. Hu, et al. (2008). Stochastic calculus for fractional Brownian motion and applications, Springer Verlag.

Cajueiro, D. O. and B. M. Tabak (2004). The Hurst exponent over time: testing the assertion that emerging markets are becoming more efficient, Physica A: Statistical Mechanics and its Applications. **336:** 521-537.

Campbell, P. and S. Abhyankar (1978). Fractals, form, chance and dimension, Springer. **1:** 35-37.

Cannon, M. J., D. B. Percival, et al. (1997). Evaluating scaled windowed variance methods for estimating the Hurst coefficient of time series, Physica A: Statistical Mechanics and its Application. **241:** 606-626.

Carmona, P. and L. Coutin (1998). Fractional Brownian motion and the Markov property, Laboratoire de statistique et probabilités, Université Paul Sabatier.

Dickey, D. A. and W. A. Fuller (1979). Distribution of the estimators for autoregressive time series with a unit root, Journal of the American Statistical Association. **74:** 427-431.

Doukhan, P., G. Oppenheim, et al. (2003). Theory and applications of long-range dependence, Birkhauser.

Ellis, C. (2007). The sampling properties of Hurst exponent estimates, Physica A: Statistical Mechanics and its Applications. **375:** 159-173.

Eom, C., S. Choi, et al. (2008). Hurst exponent and prediction based on weak-form efficient market hypothesis of stock markets, Physica A: Statistical Mechanics and its Applications. **387:** 4630-4636.

Greene, W. H. and C. Zhang (2003). Econometric analysis, Prentice Hall Upper Saddle River, NJ.

Hurst, H. E. (1951). Long-term storage capacity of reservoirs, American Society of Civil Engineers. **116:** 770-808.

Lo, A. W. and A. C. MacKinlay (2001). A non-random walk down Wall Street, Princeton University Press.

Mandelbrot, B. B. (1982). The fractal geometry of nature, Freeman.

Mandelbrot, B. B. and J. W. Van Ness (1968). Fractional Brownian motions, fractional noises and applications, SIAM Review. **10:** 422-437.

Opong, K. K., G. Mulholland, et al. (1999). The behaviour of some UK equity indices: An application of Hurst and BDS tests, Journal of Empirical Finance. **6:** 267-282.

Perron, P. and S. Ng (1996). Useful modifications to some unit root tests with dependent errors and their local asymptotic properties, Review of Economic Studies. **63:** 435-463.

Phillips, P. C. B. and P. Perron (1988). Testing for a unit root in time series regression, Biometrika. **75:** 335-346.

Pipiras, V. (2004). On the usefulness of wavelet-based simulation of fractional Brownian motion, Preprint.

Shiller, R. J. (2000). Irrational exuberance, Princeton University Press.

Stanley, H. E., L. A. N. Amaral, et al. (1999). Econophysics: Can physicists contribute to the science of economics?, Physica A: Statistical Mechanics and its Applications. **269:** 156-169.

Turvey, C. G. (2007). A note on scaled variance ratio estimation of the Hurst exponent with application to agricultural commodity prices, Physica A: Statistical Mechanics and its Applications. **377:** 155-165.

Weron, R. (2002). Estimating long-range dependence: finite sample properties and confidence intervals, Physica A: Statistical Mechanics and its Applications. **312:** 285-299.

APPENDIX A

H01: Regression and F-test statistics

Mean	101.0849
Standard Deviation	0.4825

```
stepwise, pe(0.05): regress var1 var2-var51, noconstant
                      begin with empty model
p = 0.0000 <  0.0500  adding  var2
p = 0.0000 <  0.0500  adding  var4
p = 0.0000 <  0.0500  adding  var27
p = 0.0000 <  0.0500  adding  var3
p = 0.0000 <  0.0500  adding  var7
p = 0.0008 <  0.0500  adding  var5
p = 0.0027 <  0.0500  adding  var36
p = 0.0207 <  0.0500  adding  var10

      Source |       SS       df       MS              Number of obs =    2100
-------------+------------------------------           F(  8,  2092) =       .
       Model |  44749.4407     8  5593.68008           Prob > F      =  0.0000
    Residual |  .23193045   2092  .000110865           R-squared     =  1.0000
-------------+------------------------------           Adj R-squared =  1.0000
       Total |  44749.6726  2100  21.3093679           Root MSE      =  .01053

------------------------------------------------------------------------------
        var1 |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
        var2 |    .451152    .021734    20.76   0.000     .4085294    .4937745
        var4 |   .0970622   .0240157     4.04   0.000     .0499651    .1441594
       var27 |   .0660737   .0160482     4.12   0.000     .0346016    .0975458
        var3 |   .1632168    .023795     6.86   0.000     .1165524    .2098812
        var7 |   .0701354   .0210573     3.33   0.001       .02884    .1114308
        var5 |   .0664025   .0229683     2.89   0.004     .0213594    .1114456
       var36 |   .0421749   .0156067     2.70   0.007     .0115687    .0727812
       var10 |   .0438016   .0189126     2.32   0.021     .0067122     .080891
------------------------------------------------------------------------------

.

. test var2+var4+var27+var3+var7+var5+var36+var10=1, coef

 ( 1)  var2 + var4 + var27 + var3 + var7 + var5 + var36 + var10 = 1

       F(  1,  2092) =    0.15
            Prob > F =    0.7011

Constrained coefficients

------------------------------------------------------------------------------
             |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
        var2 |   .4512212   .0217333    20.76   0.000     .4086247    .4938176
        var4 |   .0970806   .0240156     4.04   0.000     .0500108    .1441504
       var27 |   .0660354   .0160479     4.11   0.000     .0345821    .0974888
        var3 |   .1632406   .0237949     6.86   0.000     .1166034    .2098778
        var7 |   .0701448   .0210573     3.33   0.001     .0288734    .1114163
        var5 |   .0664158   .0229683     2.89   0.004     .0213989    .1114328
       var36 |   .0420861    .015605     2.70   0.007      .011501    .0726713
       var10 |   .0437754   .0189124     2.31   0.021     .0067077    .0808431
------------------------------------------------------------------------------
```

H02: Regression and F-test statistics

Mean	102.5700
Standard Deviation	0.3739

```
stepwise, pe(0.05): regress var1 var2-var51, noconstant
                      begin with empty model
p = 0.0000 <  0.0500  adding  var2
p = 0.0000 <  0.0500  adding  var4
p = 0.0000 <  0.0500  adding  var27
p = 0.0000 <  0.0500  adding  var3
p = 0.0000 <  0.0500  adding  var7
p = 0.0252 <  0.0500  adding  var5

      Source |       SS       df       MS              Number of obs =    2100
-------------+------------------------------           F(  6,  2094) =       .
       Model |  45037.1638     6  7506.19396           Prob > F      =  0.0000
    Residual |  .139528239  2094  .000066632           R-squared     =  1.0000
-------------+------------------------------           Adj R-squared =  1.0000
       Total |  45037.3033  2100  21.4463349           Root MSE      =  .00816

------------------------------------------------------------------------------
        var1 |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
        var2 |   .5805454   .0217432    26.70   0.000      .537905    .6231859
        var4 |   .0820284   .0253033     3.24   0.001     .0324062    .1316506
       var27 |   .0636792   .0106892     5.96   0.000     .0427167    .0846417
        var3 |   .1593396   .0251398     6.34   0.000      .110038    .2086413
        var7 |   .0614952   .0190355     3.23   0.001     .0241648    .0988256
        var5 |    .052931   .0236374     2.24   0.025     .0065757    .0992864
------------------------------------------------------------------------------

. test var2+var4+var27+var3+var7+var5=1, coef

 ( 1)  var2 + var4 + var27 + var3 + var7 + var5 = 1

       F(  1,  2094) =    0.24
            Prob > F =    0.6238

Constrained coefficients

------------------------------------------------------------------------------
             |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
        var2 |   .5806543    .021742    26.71   0.000     .5380407    .6232679
        var4 |   .0820462   .0253033     3.24   0.001     .0324527    .1316396
       var27 |   .0635299   .0106848     5.95   0.000      .042588    .0844718
        var3 |   .1593668   .0251398     6.34   0.000     .1100938    .2086398
        var7 |   .0614715   .0190354     3.23   0.001     .0241628    .0987803
        var5 |   .0529313   .0236374     2.24   0.025     .0066028    .0992599
------------------------------------------------------------------------------
```

H03: Regression and F-test statistics

Mean	105.0809
Standard Deviation	0.2937

```
stepwise, pe(0.05): regress var1 var2-var51, noconstant
                      begin with empty model
p = 0.0000 <  0.0500  adding  var2
p = 0.0000 <  0.0500  adding  var3
p = 0.0000 <  0.0500  adding  var27
p = 0.0000 <  0.0500  adding  var4
p = 0.0121 <  0.0500  adding  var7

      Source |       SS       df       MS              Number of obs =    2100
-------------+------------------------------           F(  5,  2095) =       .
       Model |  45515.7679     5  9103.15358           Prob > F      =  0.0000
    Residual |  .086098218  2095  .000041097           R-squared     =  1.0000
-------------+------------------------------           Adj R-squared =  1.0000
       Total |   45515.854  2100  21.6742162           Root MSE      =  .00641

------------------------------------------------------------------------------
        var1 |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
        var2 |   .7117915   .0217609    32.71   0.000     .6691163    .7544668
        var3 |   .1362037    .026659     5.11   0.000     .0839228    .1884847
       var27 |   .0381384   .0082872     4.60   0.000     .0218864    .0543904
        var4 |   .0731757   .0237319     3.08   0.002     .0266352    .1197162
        var7 |    .040712    .016213     2.51   0.012     .0089166    .0725073
------------------------------------------------------------------------------

. test var2+var3+var27+var4+var7=1, coef

 ( 1)  var2 + var3 + var27 + var4 + var7 = 1

       F(  1,  2095) =    0.50
            Prob > F =    0.4776

Constrained coefficients

------------------------------------------------------------------------------
             |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
        var2 |   .7120234   .0217585    32.72   0.000     .6693776    .7546692
        var3 |   .1362426    .026659     5.11   0.000      .083992    .1884933
       var27 |   .0378728   .0082787     4.57   0.000     .0216468    .0540989
        var4 |   .0731965   .0237319     3.08   0.002      .026683    .1197101
        var7 |   .0406646   .0162129     2.51   0.012     .0088879    .0724413
------------------------------------------------------------------------------
```

H04: Regression and F-test statistics

Mean	109.3273
Standard Deviation	0.2328

```
stepwise, pe(0.05): regress var1 var2-var51, noconstant
                      begin with empty model
p = 0.0000 <  0.0500  adding  var2
p = 0.0000 <  0.0500  adding  var3
p = 0.0001 <  0.0500  adding  var27
p = 0.0402 <  0.0500  adding  var31
p = 0.0342 <  0.0500  adding  var4

      Source |       SS       df       MS              Number of obs =    2100
-------------+------------------------------           F(  5,  2095) =       .
       Model |  46303.0521     5  9260.61043           Prob > F      =  0.0000
    Residual |  .054093137  2095  .00002582           R-squared     =  1.0000
-------------+------------------------------           Adj R-squared =  1.0000
       Total |  46303.1062  2100  22.0490982           Root MSE      =  .00508

------------------------------------------------------------------------------
        var1 |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
        var2 |   .8439946   .0217822    38.75   0.000     .8012777    .8867115
        var3 |   .0906091   .0284568     3.18   0.001     .0348026    .1464156
       var27 |   .0452262    .013121     3.45   0.001     .0194947    .0709576
       var31 |  -.0265485   .0125222    -2.12   0.034    -.0511058   -.0019913
        var4 |   .0467405   .0220594     2.12   0.034     .0034799     .090001
------------------------------------------------------------------------------


. test var2+var3+var27+var31+var4=1, coef

 ( 1)  var2 + var3 + var27 + var31 + var4 = 1

       F(  1,  2095) =    0.85
            Prob > F =    0.3575

Constrained coefficients

------------------------------------------------------------------------------
             |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
        var2 |    .844413   .0217774    38.77   0.000     .8017301     .887096
        var3 |   .0906416   .0284567     3.19   0.001     .0348674    .1464158
       var27 |    .045195   .0131209     3.44   0.001     .0194785    .0709115
       var31 |  -.0269614   .0125142    -2.15   0.031    -.0514887   -.0024341
        var4 |   .0467118   .0220593     2.12   0.034     .0034762    .0899473
------------------------------------------------------------------------------
```

H05: Regression and F-test statistics

Mean	83.4242
Standard Deviation	0.1847

```
stepwise, pe(0.05): regress var1 var2-var51, noconstant
                      begin with empty model
p = 0.0000 <  0.0500  adding  var2

      Source |       SS       df       MS              Number of obs =    2100
-------------+------------------------------           F(  1,  2099) =       .
       Model |  41009.0756     1  41009.0756           Prob > F      =  0.0000
    Residual |  .034059253  2099  .000016226           R-squared     =  1.0000
-------------+------------------------------           Adj R-squared =  1.0000
       Total |  41009.1097  2100  19.5281475           Root MSE      =  .00403

------------------------------------------------------------------------------
        var1 |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
        var2 |   .9999784   .0000199       .    0.000     .9999394    1.000017
------------------------------------------------------------------------------

. test var2=1, coef

 ( 1)  var2 = 1

       F(  1,  2099) =    1.18
            Prob > F =    0.2782

Constrained coefficients

------------------------------------------------------------------------------
             |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
        var2 |          1          .       .       .            .           .
------------------------------------------------------------------------------
```

H06: Regression and F-test statistics

Mean	129.5964
Standard Deviation	0.1494

```
stepwise, pe(0.05): regress var1 var2-var51, noconstant
                      begin with empty model
p = 0.0000 <  0.0500  adding  var2
p = 0.0000 <  0.0500  adding  var9
p = 0.0015 <  0.0500  adding  var3

      Source |       SS       df       MS              Number of obs =    2100
-------------+------------------------------           F(  3,  2097) =       .
       Model |  49710.4503     3  16570.1501           Prob > F      =  0.0000
    Residual |  .022334008  2097   .00001065           R-squared     =  1.0000
-------------+------------------------------           Adj R-squared =  1.0000
       Total |  49710.4726  2100  23.6716536           Root MSE      =  .00326

------------------------------------------------------------------------------
        var1 |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
        var2 |   1.110336   .0217804    50.98   0.000     1.067622    1.153049
        var9 |  -.0338645   .0074572    -4.54   0.000    -.0484888   -.0192402
        var3 |  -.0764468   .0239991    -3.19   0.001    -.1235114   -.0293822
------------------------------------------------------------------------------

. test var2+var9+var3=1, coef

 ( 1)  var2 + var9 + var3 = 1

       F(  1,  2097) =    2.78
            Prob > F =    0.0957

Constrained coefficients

------------------------------------------------------------------------------
             |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
        var2 |   1.111717   .0217647    51.08   0.000     1.069059    1.154375
        var9 |  -.0351105   .0074196    -4.73   0.000    -.0496527   -.0205682
        var3 |  -.0766067    .023999    -3.19   0.001    -.1236437   -.0295696
------------------------------------------------------------------------------
```

H07: Regression and F-test statistics

Mean	154.1644
Standard Deviation	0.1205

```
stepwise, pe(0.05): regress var1 var2-var51, noconstant
                      begin with empty model
p = 0.0000 <  0.0500  adding  var2
p = 0.0000 <  0.0500  adding  var4
p = 0.0000 <  0.0500  adding  var9
p = 0.0000 <  0.0500  adding  var3
p = 0.0174 <  0.0500  adding  var51

      Source |       SS       df       MS              Number of obs =    2100
-------------+------------------------------           F(  5,  2095) =       .
       Model |  53211.2799     5   10642.256           Prob > F      =  0.0000
    Residual |  .014489909  2095  6.9164e-06           R-squared     =  1.0000
-------------+------------------------------           Adj R-squared =  1.0000
       Total |  53211.2944  2100  25.3387116           Root MSE      =  .00263

------------------------------------------------------------------------------
        var1 |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
        var2 |   1.238719   .0218315    56.74   0.000     1.195905    1.281532
        var4 |  -.0387849   .0248607    -1.56   0.119     -.087539    .0099693
        var9 |  -.0333682   .0078072    -4.27   0.000    -.0486788   -.0180576
        var3 |  -.1627038   .0346494    -4.70   0.000    -.2306546   -.0947529
       var51 |  -.0038426    .001615    -2.38   0.017    -.0070098   -.0006754
------------------------------------------------------------------------------


. test var2+var4+var9+var3+var51=1, coef

 ( 1)  var2 + var4 + var9 + var3 + var51 = 1

       F(  1,  2095) =    2.57
            Prob > F =    0.1092

Constrained coefficients

------------------------------------------------------------------------------
             |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
        var2 |   1.239981   .0218173    56.83   0.000      1.19722    1.282742
        var4 |    -.03901   .0248603    -1.57   0.117    -.0877352    .0097152
        var9 |  -.0334447    .007807    -4.28   0.000    -.0487462   -.0181432
        var3 |  -.1630198   .0346488    -4.70   0.000    -.2309302   -.0951093
       var51 |  -.0045068   .0015609    -2.89   0.004    -.0075662   -.0014475
------------------------------------------------------------------------------
```

H08: Regression and F-test statistics

Mean	206.0310
Standard Deviation	0.0976

```
stepwise, pe(0.05): regress var1 var2-var51, noconstant
                      begin with empty model
p = 0.0000 <  0.0500  adding  var2
p = 0.0000 <  0.0500  adding  var3
p = 0.0000 <  0.0500  adding  var9
p = 0.0011 <  0.0500  adding  var51
p = 0.0164 <  0.0500  adding  var5

      Source |       SS       df       MS              Number of obs =    2100
-------------+------------------------------           F(  5,  2095) =       .
       Model |  59083.3218     5  11816.6644           Prob > F      =  0.0000
    Residual |  .009465085  2095  4.5179e-06           R-squared     =  1.0000
-------------+------------------------------           Adj R-squared =  1.0000
       Total |  59083.3312  2100  28.1349196           Root MSE      =  .00213

------------------------------------------------------------------------------
        var1 |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
        var2 |   1.368253   .0215345    63.54   0.000     1.326022    1.410484
        var3 |  -.2930111   .0293409    -9.99   0.000    -.3505515   -.2354706
        var9 |  -.0289155   .0079491    -3.64   0.000    -.0445045   -.0133265
       var51 |  -.0037623   .0011113    -3.39   0.001    -.0059416    -.001583
        var5 |  -.0425476   .0177182    -2.40   0.016    -.0772947   -.0078005
------------------------------------------------------------------------------


. test var2+var3+var9+var51+var5=1, coef

 ( 1)  var2 + var3 + var9 + var51 + var5 = 1

       F(  1,  2095) =    3.11
            Prob > F =    0.0780

Constrained coefficients

------------------------------------------------------------------------------
             |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
        var2 |   1.369781    .021517    63.66   0.000     1.327608    1.411954
        var3 |  -.2936669   .0293386   -10.01   0.000    -.3511695   -.2361643
        var9 |  -.0289473   .0079491    -3.64   0.000    -.0445272   -.0133673
       var51 |  -.0043476   .0010605    -4.10   0.000    -.0064262    -.002269
        var5 |  -.0428192   .0177175    -2.42   0.016    -.0775449   -.0080935
------------------------------------------------------------------------------
```

H09: Regression and F-test statistics

Mean	337.8527
Standard Deviation	0.0788

```
stepwise, pe(0.05): regress var1 var2-var51, noconstant
                      begin with empty model
p = 0.0000 <  0.0500  adding  var2
p = 0.0000 <  0.0500  adding  var3
p = 0.0000 <  0.0500  adding  var9
p = 0.0001 <  0.0500  adding  var47
p = 0.0092 <  0.0500  adding  var5

      Source |       SS       df       MS              Number of obs =    2100
-------------+------------------------------           F(  5,  2095) =       .
       Model |  69166.3233     5  13833.2647           Prob > F      =  0.0000
    Residual |  .006216057  2095  2.9671e-06           R-squared     =  1.0000
-------------+------------------------------           Adj R-squared =  1.0000
       Total |  69166.3295  2100  32.9363474           Root MSE      =  .00172

------------------------------------------------------------------------------
        var1 |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
        var2 |   1.494504   .0212558    70.31   0.000      1.45282    1.536189
        var3 |  -.4201071   .0299151   -14.04   0.000    -.4787735   -.3614408
        var9 |  -.0258653    .007082    -3.65   0.000    -.0397537   -.0119769
       var47 |  -.0035474   .0008365    -4.24   0.000    -.0051878    -.001907
        var5 |   -.044971   .0172525    -2.61   0.009    -.0788048   -.0111373
------------------------------------------------------------------------------

. test var2+var3+var9+var47+var5=1, coef

 ( 1)  var2 + var3 + var9 + var47 + var5 = 1

       F(  1,  2095) =    3.29
            Prob > F =    0.0699

Constrained coefficients

------------------------------------------------------------------------------
             |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
        var2 |   1.496118   .0212371    70.45   0.000     1.454494    1.537742
        var3 |  -.4209776   .0299112   -14.07   0.000    -.4796025   -.3623527
        var9 |  -.0258459   .0070819    -3.65   0.000    -.0397263   -.0119656
       var47 |  -.0040102   .0007966    -5.03   0.000    -.0055715   -.0024489
        var5 |  -.0452846   .0172516    -2.62   0.009    -.0790971   -.0114721
------------------------------------------------------------------------------
```

APPENDIX B

Live Cattle: Regression and F-test statistics

Mean	65.6121
Standard Deviation	0.3523

```
stepwise, pe(0.05): regress var1 var2-var51, noconstant
                      begin with empty model
p = 0.0000 <  0.0500  adding  var2
p = 0.0006 <  0.0500  adding  var50
p = 0.0000 <  0.0500  adding  var49
p = 0.0139 <  0.0500  adding  var8
p = 0.0091 <  0.0500  adding  var5

      Source |       SS       df       MS              Number of obs =     902
-------------+------------------------------           F(  5,   897) =       .
       Model |    15798.36     5  3159.67199           Prob > F      =  0.0000
    Residual |  .123443586   897  .000137618           R-squared     =  1.0000
-------------+------------------------------           Adj R-squared =  1.0000
       Total |  15798.4834   902  17.5149483           Root MSE      =  .01173

------------------------------------------------------------------------------
        var1 |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
        var2 |   .9832064   .0189653    51.84   0.000     .9459849    1.020428
       var50 |   .1393974   .0288061     4.84   0.000     .0828623    .1959325
       var49 |  -.1255901   .0290249    -4.33   0.000    -.1825547   -.0686254
        var8 |    .069417   .0193997     3.58   0.000      .031343    .1074911
        var5 |  -.0663778   .0254109    -2.61   0.009    -.1162496   -.0165059
------------------------------------------------------------------------------


. test var2+var50+var49+var8+var5=1, coef

 ( 1)  var2 + var50 + var49 + var8 + var5 = 1

       F(  1,   897) =    0.32
            Prob > F =    0.5722

Constrained coefficients

------------------------------------------------------------------------------
             |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
        var2 |   .9834966   .0189584    51.88   0.000     .9463389    1.020654
       var50 |   .1391358   .0288023     4.83   0.000     .0826842    .1955873
       var49 |  -.1256954   .0290243    -4.33   0.000     -.182582   -.0688087
        var8 |    .069461   .0193995     3.58   0.000     .0314387    .1074834
        var5 |   -.066398   .0254109    -2.61   0.009    -.1162025   -.0165935
------------------------------------------------------------------------------
```

Wheat: Regression and F-test statistics

Mean	345.3745
Standard Deviation	0.6448

```
stepwise, pe(0.05): regress var1 var2-var51, noconstant
                      begin with empty model
p = 0.0000 <  0.0500  adding  var2
p = 0.0000 <  0.0500  adding  var50
p = 0.0000 <  0.0500  adding  var49
p = 0.0170 <  0.0500  adding  var4
p = 0.0128 <  0.0500  adding  var5
p = 0.0149 <  0.0500  adding  var7
p = 0.0190 <  0.0500  adding  var10
p = 0.0341 <  0.0500  adding  var29
```

Source	SS	df	MS
Model	30237.9681	8	3779.74601
Residual	.412169598	894	.00046104
Total	30238.3803	902	33.5237032

```
Number of obs =      902
F(  8,   894) =        .
Prob > F      =   0.0000
R-squared     =   1.0000
Adj R-squared =   1.0000
Root MSE      =   .02147
```

var1	Coef.	Std. Err.	t	P>|t|	[95% Conf.	Interval]
var2	.9236606	.0232469	39.73	0.000	.8780358	.9692854
var50	.2424556	.028008	8.66	0.000	.1874865	.2974247
var49	-.2275638	.0283679	-8.02	0.000	-.2832392	-.1718885
var4	.1236448	.037424	3.30	0.001	.0501956	.1970941
var5	-.1306025	.0374823	-3.48	0.001	-.2041661	-.0570389
var7	.1008222	.029292	3.44	0.001	.0433331	.1583113
var10	-.0551437	.0204542	-2.70	0.007	-.0952875	-.0149998
var29	.0223835	.0105452	2.12	0.034	.0016874	.0430797

```
. test var2+var50+var49+var4+var5+var7+var10+var29=1, coef

 ( 1)  var2 + var50 + var49 + var4 + var5 + var7 + var10 + var29 = 1

       F(  1,   894) =    6.67
            Prob > F =    0.0099
```

Constrained coefficients

	Coef.	Std. Err.	z	P>|z|	[95% Conf.	Interval]
var2	.9291032	.0231512	40.13	0.000	.8837277	.9744787
var50	.2375285	.027943	8.50	0.000	.1827612	.2922958
var49	-.2296391	.0283565	-8.10	0.000	-.2852168	-.1740614
var4	.1237614	.037424	3.31	0.001	.0504116	.1971111
var5	-.1307927	.0374823	-3.49	0.000	-.2042565	-.0573288
var7	.1015567	.0292906	3.47	0.001	.0441481	.1589653
var10	-.0536245	.0204458	-2.62	0.009	-.0936974	-.0135515
var29	.0221065	.0105446	2.10	0.036	.0014394	.0427735

Lean Hogs: Regression and F-test statistics

Mean	60.2496
Standard Deviation	0.7938

```
stepwise, pe(0.05): regress var1 var2-var51, noconstant
                      begin with empty model
p = 0.0000 <  0.0500  adding  var2

      Source |       SS       df       MS              Number of obs =     902
-------------+------------------------------           F(  1,   901) =       .
       Model |  14911.7034     1  14911.7034           Prob > F      =  0.0000
    Residual |  .629392387   901  .000698549           R-squared     =  1.0000
-------------+------------------------------           Adj R-squared =  1.0000
       Total |  14912.3328   902  16.5325197           Root MSE      =  .02643

------------------------------------------------------------------------------
        var1 |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
        var2 |   .9999438   .0002164  4620.25   0.000     .9995191    1.000369
------------------------------------------------------------------------------

. test var2=1, coef

 ( 1)  var2 = 1

       F(  1,   901) =    0.07
            Prob > F =    0.7953

Constrained coefficients

------------------------------------------------------------------------------
             |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
        var2 |          1   2.57e-12       .    0.000            1           1
------------------------------------------------------------------------------
```

Pork Bellies: Regression and F-test statistics

Mean	63.9827
Standard Deviation	0.9989

```
stepwise, pe(0.05): regress var1 var2-var51, noconstant
                      begin with empty model
p = 0.0000 <  0.0500  adding  var2
p = 0.0476 <  0.0500  adding  var8

      Source |       SS       df       MS              Number of obs =     902
-------------+------------------------------           F(  2,   900) =       .
       Model |  15408.0553     2  7704.02765           Prob > F      =  0.0000
    Residual |  .99583295    900  .001106481           R-squared     =  0.9999
-------------+------------------------------           Adj R-squared =  0.9999
       Total |  15409.0511   902  17.0832053           Root MSE      =  .03326

------------------------------------------------------------------------------
        var1 |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
        var2 |   1.025448   .0127924    80.16   0.000     1.000341    1.050554
        var8 |  -.0253986   .0128019    -1.98   0.048    -.0505236   -.0002736
------------------------------------------------------------------------------


. test var2+var8=1, coef

 ( 1)  var2 + var8 = 1

       F(  1,   900) =    0.03
            Prob > F =    0.8547

Constrained coefficients

------------------------------------------------------------------------------
             |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
        var2 |   1.025506   .0127885    80.19   0.000     1.000441    1.050571
        var8 |  -.025506    .0127885    -1.99   0.046    -.0505709   -.000441
------------------------------------------------------------------------------
```

Coffee: Regression and F-test statistics

Mean	129.1575
Standard Deviation	0.9839

```
stepwise, pe(0.05): regress var1 var2-var51, noconstant
                      begin with empty model
p = 0.0000 <  0.0500  adding  var2
p = 0.0071 <  0.0500  adding  var32
p = 0.0108 <  0.0500  adding  var34
p = 0.0249 <  0.0500  adding  var30

      Source |       SS       df       MS              Number of obs =     902
-------------+------------------------------           F(  4,   898) =       .
       Model |  21164.7248     4  5291.18121           Prob > F      =  0.0000
    Residual |  .963520685   898  .001072963           R-squared     =  1.0000
-------------+------------------------------           Adj R-squared =  1.0000
       Total |  21165.6884   902  23.4652864           Root MSE      =  .03276

------------------------------------------------------------------------------
        var1 |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
        var2 |   .9860796   .0076263   129.30   0.000     .9711122    1.001047
       var32 |   .1251617   .0318008     3.94   0.000     .0627492    .1875743
       var34 |  -.0590013   .0228202    -2.59   0.010    -.1037884   -.0142141
       var30 |  -.0522907   .0232724    -2.25   0.025    -.0979652   -.0066161
------------------------------------------------------------------------------

. test var2+var32+var34+var30=1, coef

 ( 1)  var2 + var32 + var34 + var30 = 1

       F(  1,   898) =    0.05
            Prob > F =    0.8225

Constrained coefficients

------------------------------------------------------------------------------
             |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
        var2 |    .986107   .0076253   129.32   0.000     .9711615    1.001052
       var32 |   .1251542   .0318008     3.94   0.000     .0628258    .1874825
       var34 |  -.0589844   .0228201    -2.58   0.010    -.1037109   -.0142578
       var30 |  -.0522767   .0232723    -2.25   0.025    -.0978896   -.0066639
------------------------------------------------------------------------------
```

CHAPTER III

AN EXCURSION INTO FRACTAL SPACE

1. Introduction

The Efficient Market Hypothesis (EMH) has been the core structure of financial economics theory for decades. EMH states that current market prices of securities fully reflect all available information and that markets are orderly and tidy. Moreover, investors always act in a rational way (Fama 1970; Lo 2007). However, recent studies have found that the markets as we know from experience are not orderly but instead they are complex and messy (Peters 1996; Malkiel 2003). Some people refer to this environment as *fractals*. Numerous works have shown that fractals give structure to complexity and depth to chaos; therefore, allowing us to analyze a wide range of natural phenomena as well as financial events that are difficult to explain by the EMH from a new perspective (Mandelbrot 1982; Peters 1996).

The development of fractal geometry and analysis has led to the introduction of a new class of stochastic process called fractional Brownian motion (fBm) as opposed to the "random walk" process used in EMH. The fBm process is generally characterized by its associated Hurst exponent that describes the feedback effect within the process. To better understand fractals in a simple sense, we resort to the Itô's excursion theory (Rogers 1989; Pitman and Yor 2007; Watanabe 2010). This mathematical concept was created by Kiyosi Itô, the inventor of the famous Itô's Lemma used in the derivation of the Black-Scholes option pricing formula (Black and Scholes 1973; Merton 1973; Turvey 2010). In the finance literature, not much attention has been given to this theory, but we will show its merits by incorporating the beauty of the Itô's excursion theory to Hurst exponent and present it as a powerful tool to analyze fractal processes.

In this paper, we revisit this theory of excursion point process and apply the technique to detect and analyze the behavior of a known fractional Brownian motion. We demonstrate the key attributes and benefits of this tool via a step-by-step decomposition of the theory into simple mathematical terms. After that, we construct a program to measure excursion variables such as local time and excursion length based on such algorithm. This will be a discrete-time application of the theory. Then, with a known set of data that behaves as a pure Brownian motion, we capture this quality using the Itô's excursion theory and show that the excursions-valued process actually follows a binomial distribution which is a robust substitute for Poisson distribution as suggested from the theory (Itô 1972). The ability to replicate the mathematical concept of the excursion theory using discrete-time Monte Carlo simulation will then allow us to analyze the result observed when a known fractional Brownian motion is used as a data generator instead of the Brownian null. Hurst exponent is used as a fractal gauge for such processes. Finally, we conclude with an empirical application of the theory to an unrestricted set of commodity futures price data and again demonstrate the robustness of the method when applied to real market data.

2. Itô's excursion theory

Imagine yourself travelling through time space where every day is a brand new day. What is going to happen tomorrow is independent of the thing that has happened today.

In other words, you have no memory. This may be daunting to some but also could be very appealing to others. It may sound like a fantasy but this simple thought actually captures the essence of the Itô's excursion theory.

Itô's excursion theory is a probabilistic tool that can be applied to any continuous-time Markov process which has some recurrent state as will be outlined later in this section. It allows one to evaluate, measure, and quantify certain characteristic of the stochastic process. The theory also can be developed in great generality. The components involved in the derivation are not overly complicated, yet the power of the techniques is impressive in view of their simplicity (Rogers 1989). The clarity gained from using this method versus other techniques found in the probabilist's tool kit is enormous as we shall see it here. Despite the generality that the theory has to offer in term of its application to any continuous-time Markov process, in this study we will concentrate on one class of process called the Brownian motion. In fact, we will be extending the analysis to a more general class of Brownian motion called the fractional Brownian motion.

2.1 Symmetric random walk

In discrete time framework, the building block that makes up a Brownian motion in continuous time is the classical process called symmetric random walk. It can be described using the following example.

Consider a game where you toss a fair coin repeatedly, winning 1 each time the coin falls heads and losing 1 each time the coin falls tails. Thus, the payoff is symmetric and random. Denote X_n as your winnings on the nth toss, then X_n are independent random variables with common distribution

$$P(X_n = 1) = P(X_n = -1) = \frac{1}{2} ,$$

and your net profit and loss after playing this game n times are

$$Y_n = X_1 + X_2 + \cdots + X_n ,$$

where $n \geq 1$ and $Y_0 = 0$.

Now, define T_n as the nth time that the random walk returns to zero with initial condition $T_0 = 0$ or formally,

$$T_{n+1} = \inf\{k > T_n : Y_k = 0\} ,$$

where $n \geq 0$.

Your net profit and loss when the coin has been tossed T_n times are zero, and thus it can be seen that:

i. whatever happens after time T_n is independent of what happened before

ii. the evolution of the game from time T_n onward, $\{Y_{T_n+k}: k \geq 0\}$, is similar to the evolution of the original game $\{Y_k: k \geq 0\}$.

Therefore, if we define the *n*th excursion ξ_n by

$$\xi_n = \{Y_k: T_{n-1} \leq k \leq T_n\} ,$$

where $n \in N$, then the excursions $\xi_1, \xi_2, \ldots$ are independent and identically distributed.

Figure 1 below shows the typical path of this game.

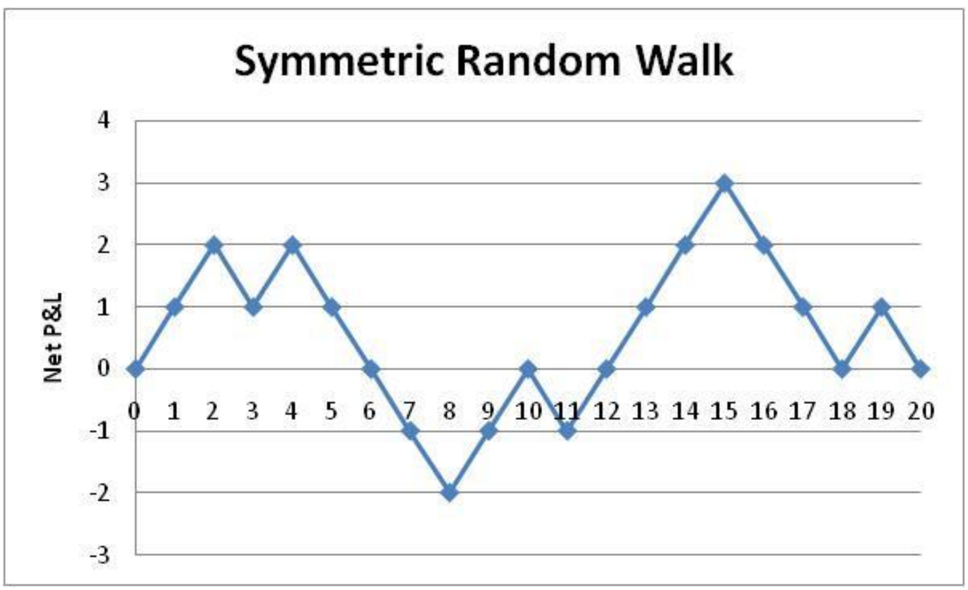


Figure 1: Symmetric random walk

Based on the sample realization in Figure 1, one can conclude that:

i. $T_0 = 0, T_1 = 6, T_2 = 10, T_3 = 12, T_4 = 18, T_5 = 20$

ii. $\xi_1 = \{Y_0, Y_1, Y_2, Y_3, Y_4, Y_5, Y_6\}$

$\xi_2 = \{Y_6, Y_7, Y_8, Y_9, Y_{10}\}$

$\xi_3 = \{Y_{10}, Y_{11}, Y_{12}\}$

$\xi_4 = \{Y_{12}, Y_{13}, Y_{14}, Y_{15}, Y_{16}, Y_{17}, Y_{18}\}$

$\xi_5 = \{Y_{18}, Y_{19}, Y_{20}\}$

2.2 Excursion measures

The key purpose of the Itô's excursion theory is to describe the evolution of a Markov process in terms of its behavior between visits to a particular point *m* in the state space. Each excursion can be thought of as a piece of random, finite path length that starts and finishes at *m*. Therefore, for a two-dimensional space as plotted in Figure 1, Itô's excursion theory can be used as a way to reconstruct the process by defining the excursion measures at many discrete level *m* and subsequently takes an infinitesimally small increment between points. This will allow one to connect all the dots that eventually form the entire path of the process (Pitman and Yor 2007).

In order to understand the core of the theory, a few important measures have to be defined and introduced. First, Itô himself introduced the concept of assigning a tag or unique identifier to each excursion by viewing the excursions as a point process indexed by local time. The local time $L_m(t) = l$ is defined as the number of visits to point *m* up to time *t*. This measure grows as *t* gets larger. Thus, it is not difficult to realize that the measure T_n and ξ_n that were introduced in Section 2.1 together define the inverse local time process. Moreover, for an excursion ξ_n spanning the time interval [T_{n-1}, T_n], the increasing process $L_m(t)$ has the same value for all $t \in [T_{n-1}, T_n]$. As such, the process of excursions could be organized by constructing a point process in which each excursion ξ_n appears at its corresponding random time interval $t \in [T_{n-1}, T_n]$.

The process of excursions is then further categorized into different "type" (γ) based on its excursion length. Itô showed that the excursion point process is a simple Poisson process with excursion measure,

$$N_l^{\gamma} = \sum_{n=1}^{l} 1_{\gamma}(\xi_n) \ ,$$

where the stopping time is set at the local time l. This measure counts the number of excursions of type γ that occurs before the local time reaches l. It is a simple Poison process because it has stationary independent increments that increase by unit jumps. The intensity of the process is defined as:

$$\boldsymbol{n}(\gamma) = \frac{1}{l} E[N_l^{\gamma}] \ .$$

Furthermore, since excursions follow a Poisson distribution, it can be easily deduced that the waiting time between two events will be exponentially distributed with mean $E\left[N_l^{\gamma}\right] = \lambda$, for a given set of parameters γ and l. Hence,

$$P(no\ excursion\ of\ type\ \gamma\ in\ (0,t) for\ a\ given\ l) = \ 1 - \left(1 - e^{-t\lambda}\right) = \ e^{-t\lambda} \ .$$

3. Relationship between excursion theory and generalized fractional Brownian motion

Itô's excursion theory has profound and solid theoretical representations but disappointingly very few applications to date. In this section, an attempt is made to empirically link the excursion theory to some practical applications via Hurst exponent, a measure that describes the degree of fractal in a generalized fractional Brownian motion process (fBm). In Section 3.1, an algorithm is built to measure and implement the excursion theory to any known set of data. The theory is verified by observing the excursion measures and distribution found using a set of data with Hurst exponent H = 0.5, which represents the case of a pure Brownian motion, as an input. This base case will serve as the benchmark for comparison against the results from other known fBm with Hurst exponent different from 0.5 that are used in Section 3.2. We state a hypothesis that a process with low Hurst exponent or short-memory process should have higher mean excursion measure at low excursion length as compared to a process with high Hurst exponent or a long-memory process. This is simply because a mean-reverting process will tend to experience shorter excursions away from a reference level m, while a long-memory process can wander away for a long period of time due to the accumulated memory. On the other hand, we will show that at high excursion length a process with high Hurst exponent should have higher mean excursion measure than a process with lower Hurst exponent. This essentially captures the fact that a longer excursion is more prominent in a long-memory process than in the case of a Brownian motion.

3.1 Brownian motion base case (H = 0.5)

As mentioned in Section 2.1, a Brownian motion process can be thought of as a continuous version of the scaled random walk obtained by taking appropriate limit. Therefore, it is not surprising that Brownian motion inherits some properties from the random walk process. In particular, we can define Brownian motion as followed (Shreve 2004).

Let $(\Omega,\mathcal{F},\mathcal{P})$ be a probability space. For each $\omega \in \Omega$, suppose there is a continuous function $W(t)$ of $t \geq 0$ that satisfies $W(0) = 0$ and that depends on ω. Then $W(t)$, $t \geq 0$, is a Brownian motion if for all $0 = t_0 < t_1 < \ldots < t_m$ the increments

$$W(t_1) = W(t_1) - W(t_0),\ W(t_2) - W(t_1), \ldots,\ W(t_m) - W(t_{m-1})$$

are independent and each of these increments is normally distributed with

$$E[W(t_{i+1}) - W(t_i)] = 0,$$

$$Var[W(t_{i+1}) - W(t_i)] = t_{i+1} - t_i.$$

It is not a coincidence that a Brownian motion process has a Hurst exponent of 0.5 as shown in (Peters 1989). This implies that Brownian motion behaves randomly without any structured memory in it.

In this section, a known, generalized fBm series with Hurst exponent equal to 0.5 is generated in Matlab using the algorithm proposed by (Abry 1996). The time series contains 2150 data points. Then, we further introduce a proprietary algorithm to replicate the known generalized fBm series using AR model such that the new time series will be consistent in nature to the original series based on their specific Hurst exponent value. In order to achieve this goal, the known time series is used to generate a set of lag series up to 50 lags. This data set serves as the input in STATA where a stepwise regression is performed to find the AR lag structure that provides the best fit to the generalized fBm series. Additionally, a test of linear restriction is implemented to ensure that any lag structure recommended actually satisfies the necessary and sufficient condition of a unit root process with the sum of the estimated coefficients equal to unity. This is shown to be true at 5% significant level.

Finally, with the AR coefficient estimators from the regression, Monte Carlo simulation is run for 30,000 iterations on the AR model with initial value being set equal to the mean of the original series and standard deviation of the random shock following the expression, $\sigma = (Root\ MSE)(\sqrt{Number\ of\ observation})$. The mean Hurst exponent at each lag is recorded and shown in Table 1 below.

Table 1: Mean Hurst exponent from simulation

Lag		5	10	20	30	35	46	50
H05	Actual H	0.50	0.51	0.50	0.50	0.50	0.50	0.50
	Simulated Mean H	0.50	0.50	0.50	0.50	0.50	0.50	0.49

As suggested by the study in (Weron 2002), the recommended lag (k) to analyze the Hurst estimator follows the relationship defined by $k = \sqrt{N}$, where N = 2150 in our study. Hence, we might want to pay particular attention to the Hurst estimation at lag k = 46.

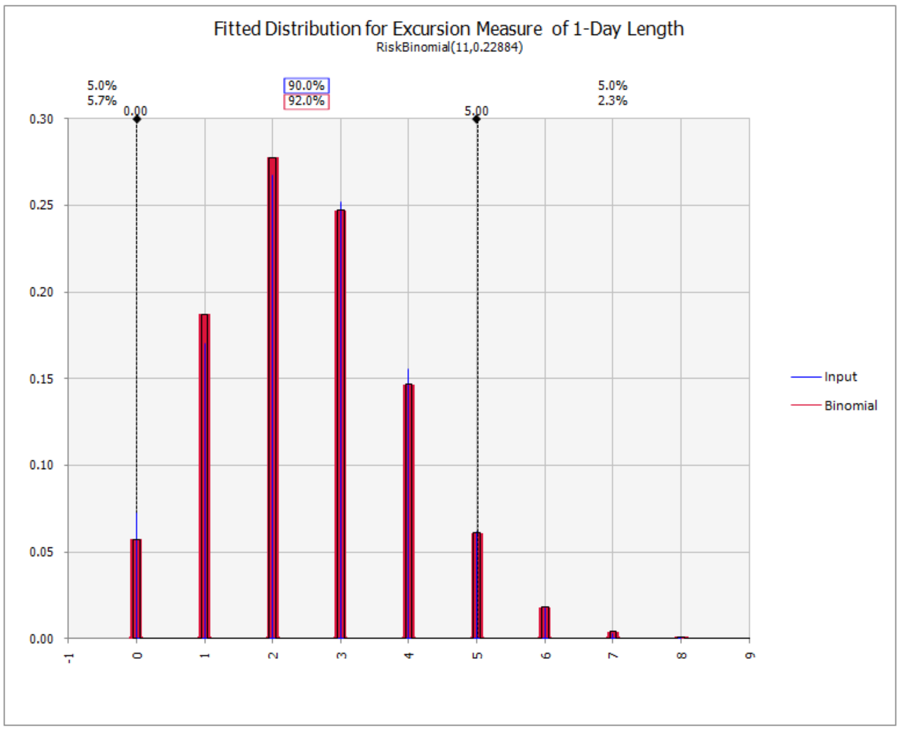
Aside from the Hurst exponent estimation, the algorithm is also coded to calculate and record several excursion statistics from the simulation as outlined in Section 2.2 of the paper. To illustrate the numerical usefulness of the Itô's excursion theory, we arbitrarily fix the stopping local time $L_m(t) = 10$, where the reference level m is set to be equal to the mean value of the time series obtained from the simulation. This procedure would later allow us to analyze the dynamics of these variables as we proceed into generating different sets of fBm in Section 3.2. Table 2 shows the descriptive statistics obtained for these variables from the simulation.

Table 2: Descriptive excursion statistics for simulated process with H = 0.5

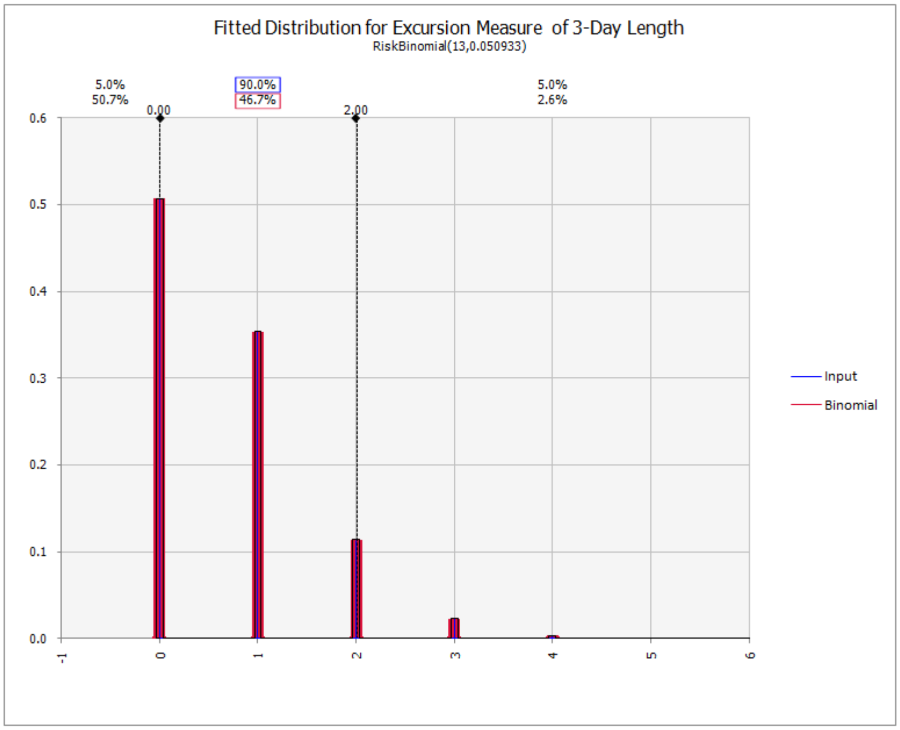
	Min	**Mean**	**Max**	**5%**	**95%**
Stopping Natural Time	18	777.7246	2149	117	1791
Overall Local Time	2	40.0516	161	6	86
Excursion Reference Level	-14.3707	4.423918	23.24594	-3.66388	12.51745

From Table 2, we could see that for a sample size of 2150 data points the process on average return to the reference level 40 times or equivalently it experiences around 40 excursions away from the reference level. The inverse local time process or stopping natural time corresponding to a fixed local time $L_m(t) = 10$ has a mean of around 778 with distribution as shown in Table 2. This implies that it takes around 778 time-steps on average to hit 10 excursions away from the mean. The excursion reference level is recorded in log scale.

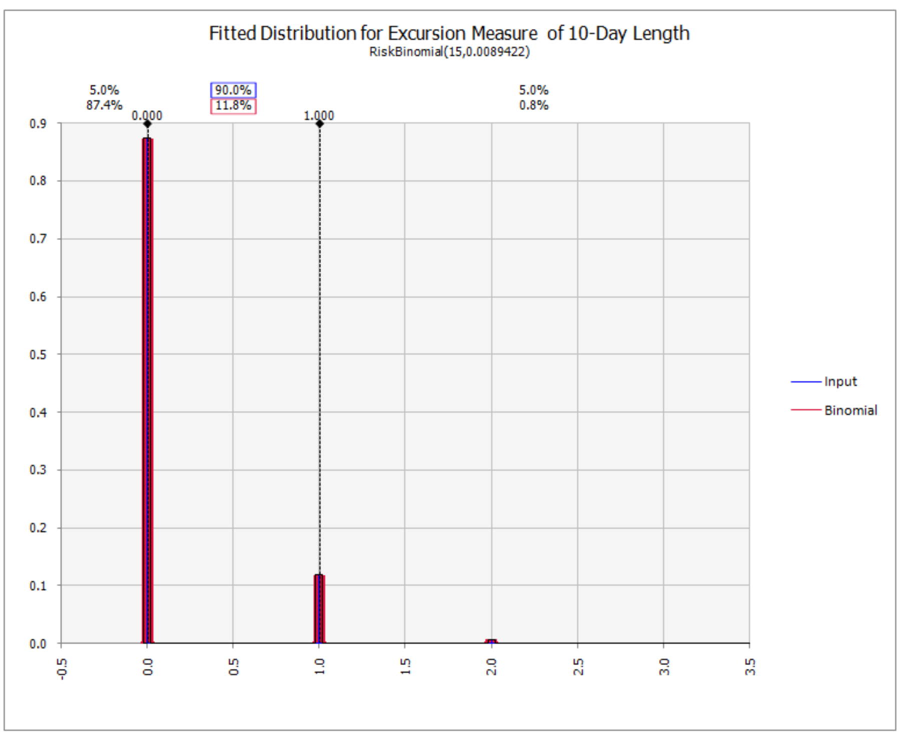
Lastly, the key excursion measures and distribution for different excursion length are collected from the simulation. In order to verify the theory using our discrete-time approximation framework, the counting process is plotted for each excursion length (γ) and fitted with the best-fit distribution based on a simple goodness-of-fit test. Figure 2 below shows some sample results for γ = 1, 3, 10, 25.



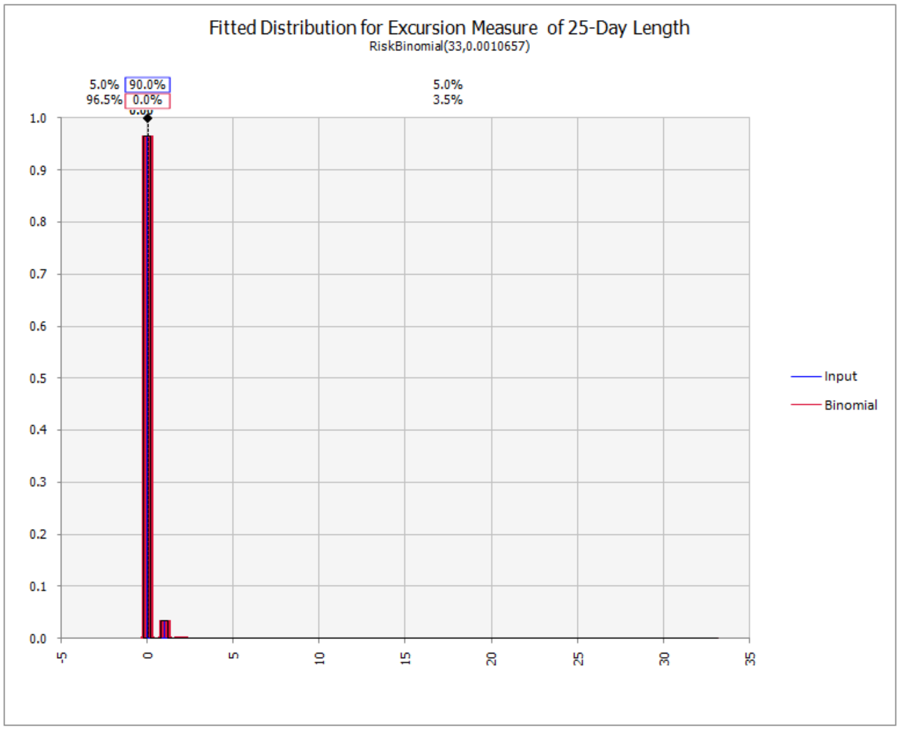
(a) γ = 1, Binomial Distribution



(b) γ = 3, Binomial Distribution



(c) γ = 10, Binomial Distribution



(d) γ = 25, Binomial Distribution

Figure 2 (a,b,c,d): Excursion measure with fitted distribution for H = 0.5

Although, the excursions-valued process is best fitted with a binomial distribution for γ = 1, 3, 10, and 25, the second-best fitted distribution is consistently shown to be Poisson with only slight difference in chi-square criterion test value. In general, we find that the results from the simulation using our discrete-time approximation framework represented the original theory very closely as shown in the consistency of the distribution of the key excursion measure in comparison to the Poisson distributed variable dictated by the theory. This confirms the usefulness and validity of our approach on the study of any further applications to the Itô's excursion theory.

3.2 Fractional Brownian motion ($H \neq 0.5$)

Fractional Brownian motion is simply an extension of the well-known Brownian motion to the fractal dimensions. It was first introduced by Kolmogorov in 1940 when it was called *Wiener Helix*. Later, Mandelbrot and Van Ness gave the process its name *fractional Brownian motion* (Mandelbrot and Van Ness 1968; Campbell and Abhyankar 1978; Mandelbrot 1982). A fBm with Hurst exponent H belonging to (0,1) is a continuous and centered Gaussian process with covariance

$$E[B_t^H B_s^H] = \frac{\sigma^2}{2}(|t|^{2H} + |s|^{2H} - |t-s|^{2H}) \qquad (s,t \geq 0),$$

where σ^2 is the variance of one-period return.

A fBm starts from zero almost surely, has stationary increments, and is self affine. For $H = 0.5$, the fBm becomes a standard Brownian motion where the increments are independent. If $H < 0.5$, the increments are negatively correlated resulting in a mean-reversion or ergodic process. When $H > 0.5$, they are positively correlated and leads to a long-memory process (Carmona and Coutin 1998; Turvey 2007).

Similar to Section 3.1, some known, generalized fBm series are generated in Matlab using the algorithm proposed by (Abry 1996) but now with Hurst exponents that are different from 0.5. Specifically, nine processes with Hurst exponent ranging from 0.1 to 0.9 at an increment of 0.1 are produced. Each of these time series also contains 2150 data points. Repeating the procedure described in Section 3.1, we are able to perform simulation on these nine different sets of data and analyze them using the Itô's excursion theory. Hurst exponent estimations for all nine processes are shown in Table 3 below.

Table 3: Actual Hurst exponent versus simulated result

Lag			5	10	20	30	35	46	50
H01	Actual	H	0.15	0.15	0.15	0.14	0.14	0.14	0.14
	Simulated	Mean H	0.16	0.16	0.16	0.14	0.14	0.14	0.15
H02	Actual	H	0.22	0.23	0.23	0.22	0.22	0.22	0.22
	Simulated	Mean H	0.23	0.24	0.23	0.22	0.22	0.22	0.23
H03	Actual	H	0.30	0.32	0.32	0.31	0.30	0.31	0.31
	Simulated	Mean H	0.31	0.32	0.31	0.30	0.30	0.30	0.30
H04	Actual	H	0.40	0.41	0.41	0.40	0.40	0.40	0.41
	Simulated	Mean H	0.40	0.41	0.40	0.39	0.39	0.40	0.40
H05	Actual	H	0.50	0.51	0.50	0.50	0.50	0.50	0.50
	Simulated	Mean H	0.50	0.50	0.50	0.50	0.50	0.50	0.49
H06	Actual	H	0.60	0.61	0.60	0.60	0.59	0.60	0.60
	Simulated	Mean H	0.60	0.61	0.60	0.59	0.59	0.59	0.58
H07	Actual	H	0.70	0.70	0.70	0.69	0.69	0.69	0.69
	Simulated	Mean H	0.72	0.72	0.72	0.72	0.71	0.71	0.71
H08	Actual	H	0.79	0.79	0.78	0.78	0.78	0.78	0.78
	Simulated	Mean H	0.86	0.85	0.84	0.83	0.83	0.83	0.83
H09	Actual	H	0.87	0.87	0.86	0.86	0.85	0.85	0.85
	Simulated	Mean H	0.97	0.97	0.95	0.94	0.94	0.94	0.94

Similarly, the stopping local time is fixed to $L_m(t) = 10$, where the reference level m is set to be equal to the mean value of each time series obtained from the simulation. The results are collected for each simulation run and are then compiled into a presentable format for further analysis. Figure 3, 4, and 5 below show the dynamics of three excursion statistics as we vary the characteristic Hurst exponent of the process generator.

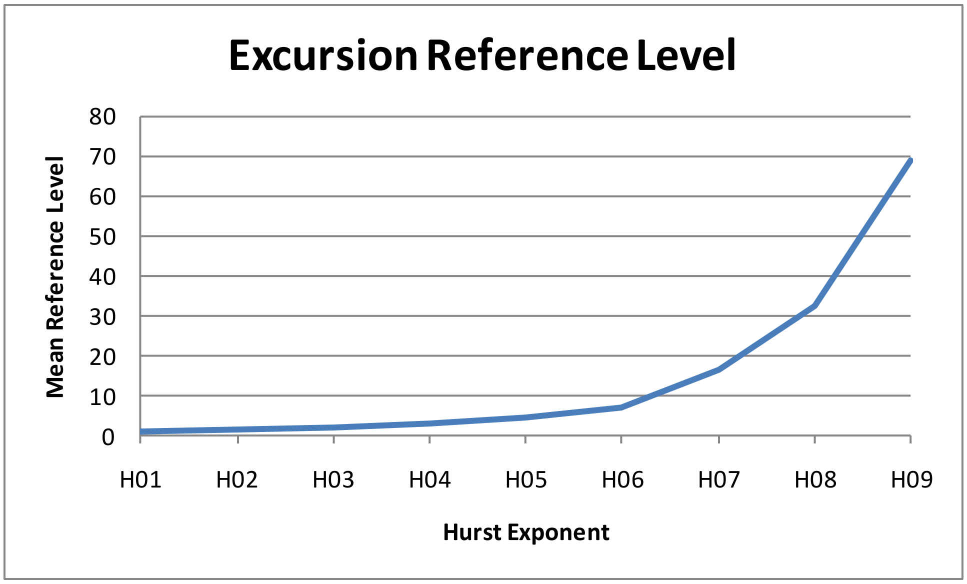


Figure 3: Dynamics of the excursion reference level

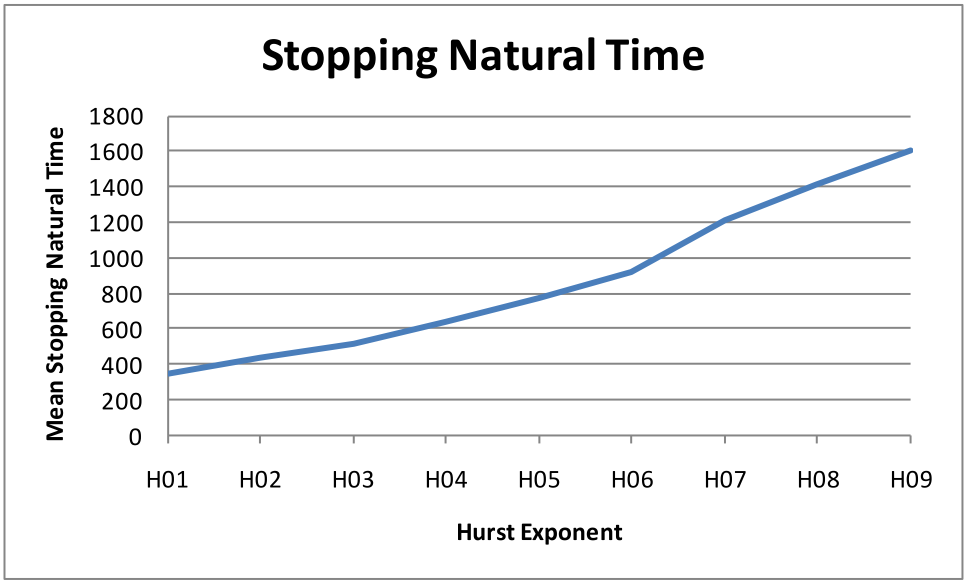


Figure 4: Dynamics of the stopping natural time

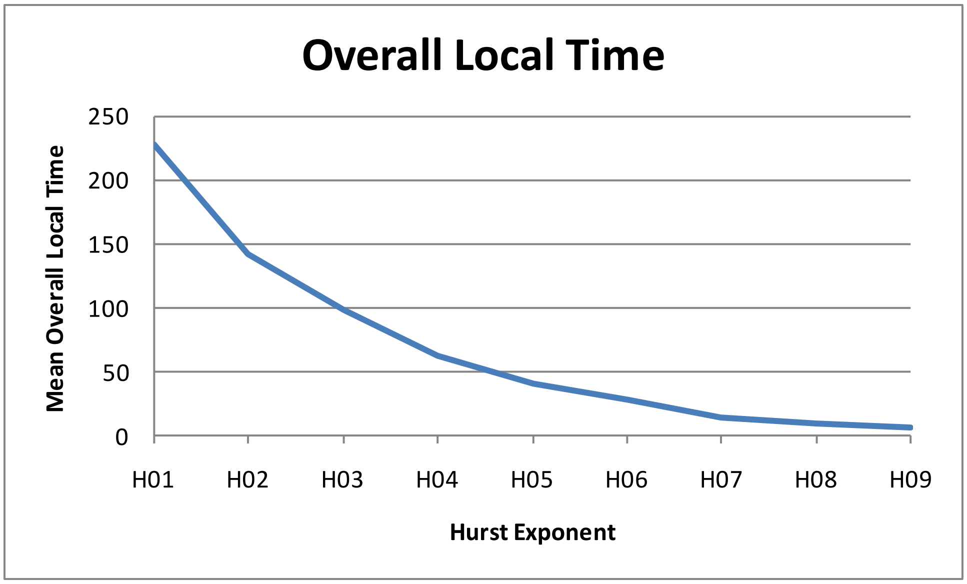


Figure 5: Dynamics of the overall local time

From Figure 3, we can see that on average the reference level increases as we increase the Hurst exponent leading to a long-memory process. This partly reveals the different characteristic observed in the path process of fBm with varying Hurst exponent from a mean-reversion process to one with long memory. It should be quite intuitive that a long-memory process with higher Hurst exponent would be less likely to experience frequent excursions as compared to those with low Hurst exponent. This is because in a long-memory process we are likely to observe systematic wandering away from the reference level due to the push factor generated from the memory accumulation through time. This leads to a downward sloping graph shown in Figure 5. Figure 4 demonstrates the nature of the inverse local time process as we vary the Hurst exponent. It confirms the understanding that the process with high Hurst exponent is likely to require more time to experience the same number of excursions away from a certain reference level.

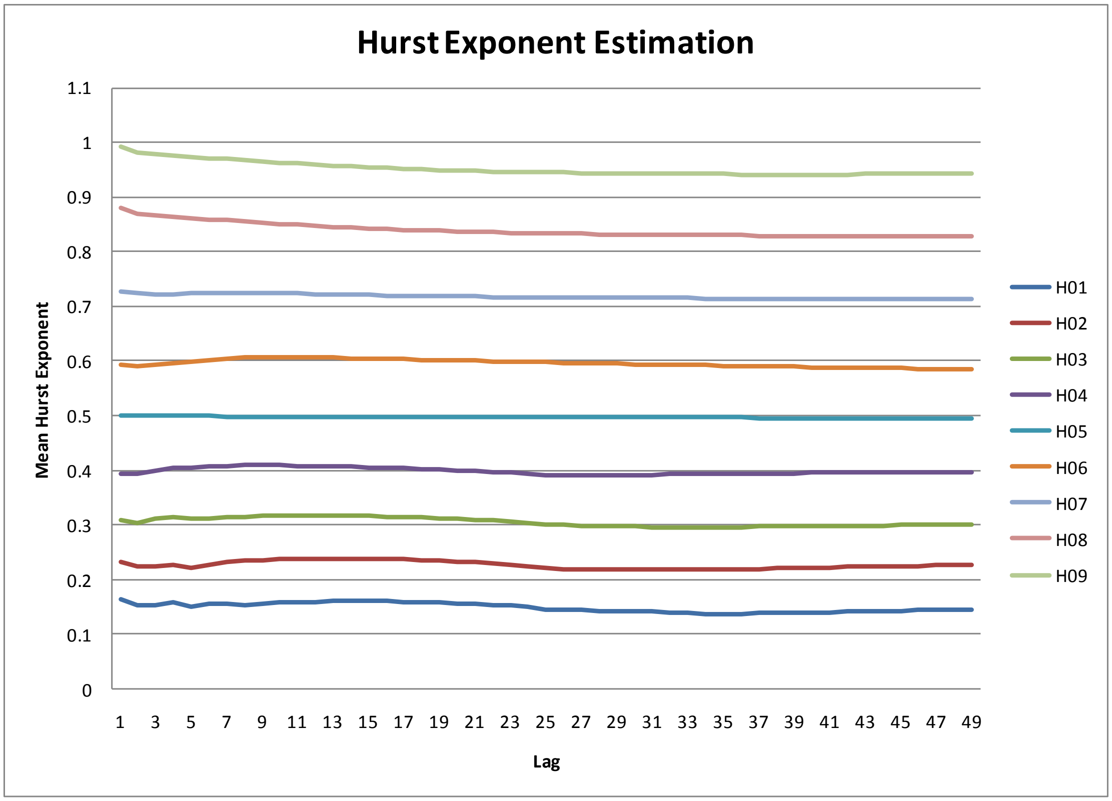


Figure 6: Hurst exponent estimation from simulation

Finally, in order to support and prove the hypothesis made earlier about the behavior of the generalized fBm process with varying Hurst exponent, we compile the resulting mean count data for each excursion length and for each of the nine processes from the simulation. Then, for each excursion length (bin), we sort for the corresponding process that achieves the highest mean excursion measure. The result is presented in Figure 7 below.

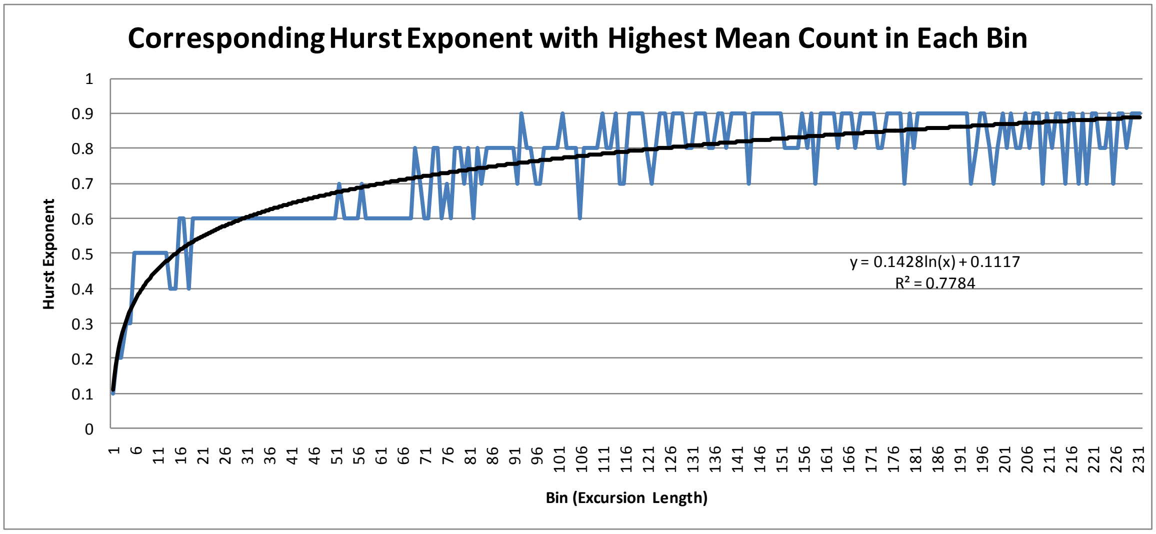


Figure 7: Relationship between Hurst exponent and mean excursion measure

It is clear from Figure 7 that a process with low Hurst exponent or short-memory process exhibits the highest mean excursion measure at low excursion length. However, as we move up the bin to one with higher excursion length, a process with higher Hurst exponent starts to gain dominance gradually. As the excursion length gets very large, the corresponding process that achieves the highest mean excursion measure tends to be those at the high end of the Hurst spectrum. This illustrates the fact that a mean-reverting process will have the tendency to experience shorter excursions away from a reference level, while a long-memory process can wander away for a long period of time due to the accumulated memory. To further support this argument, we pick a pair of fBm processes: one that is ergodic with Hurst exponent around 0.3 and the other one that has long memory with Hurst exponent around 0.7. Then, we plot the mean excursion measure against the corresponding excursion length (bin) obtained from the simulation. This is shown in Figure 8 below. We could see that initially at low excursion length the process with Hurst exponent equals 0.3 has higher mean excursion measure than the long-memory process with Hurst exponent around 0.7. Nonetheless, the scheme switches at around 'bin 28' and the 'H07' dominated the 'H03' series from that point onward as we move into bins with higher excursion length. This essentially captures the fact that a longer excursion is more prominent in a long-memory process than in the case of a mean-reverting process and vice versa.

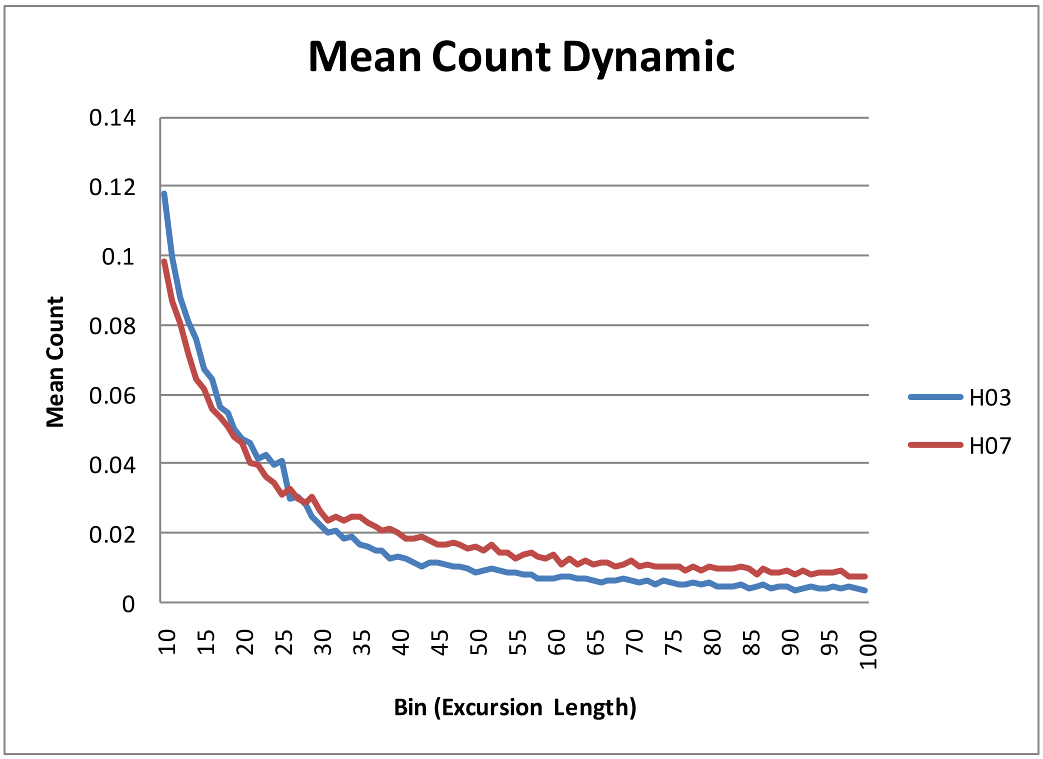


Figure 8: Mean count dynamics between ergodic and long-memory processes

4. Empirical application to commodity futures price data

Having established the benchmark results using synthetic fBm data set, we will now extend our analysis to real market data where the persistency in the time series is not known beforehand. We again use the same set of commodity futures price data from the previous study. The method described in Section 3.1 is replicated using this new data set of five commodity futures price time series as inputs. We then perform Monte Carlo simulation on the structured model for 30,000 iterations and analyze the results using the Itô's excursion theory. Hurst exponent estimations are shown in Table 4 and Figure 9 below.

Table 4: Actual Hurst exponent versus simulated result for commodity futures price data

Lag			**5**	**10**	**15**	**20**	**31**	**40**	**50**
Live Cattle	Actual	H	0.47	0.42	0.41	0.40	0.43	0.43	0.41
	Simulated	Mean H	0.49	0.44	0.43	0.42	0.41	0.41	0.39
Wheat	Actual	H	0.45	0.44	0.43	0.41	0.40	0.39	0.38
	Simulated	Mean H	0.45	0.45	0.44	0.43	0.40	0.39	0.39
Lean Hogs	Actual	H	0.48	0.50	0.51	0.51	0.52	0.51	0.50
	Simulated	Mean H	0.50	0.50	0.50	0.49	0.49	0.49	0.49
Pork Bellies	Actual	H	0.53	0.53	0.53	0.53	0.53	0.51	0.50
	Simulated	Mean H	0.53	0.54	0.54	0.54	0.53	0.53	0.53
Coffee	Actual	H	0.49	0.47	0.45	0.45	0.44	0.43	0.44
	Simulated	Mean H	0.48	0.48	0.47	0.46	0.44	0.43	0.43

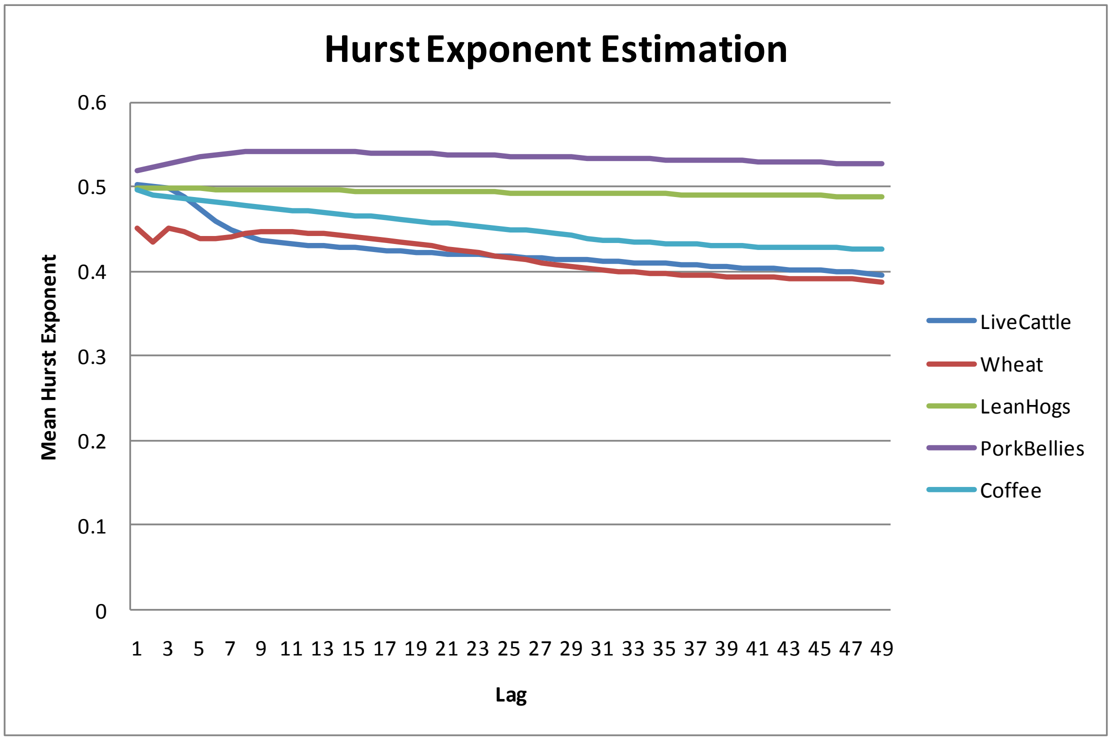


Figure 9: Hurst exponent estimation from simulation for commodity futures price data

Since the data set is now smaller and only contain N = 952 sample points, we would instead be interested in the Hurst estimation at lag k = 31 as specified by the rule, $k = \sqrt{N}$. The comparison between the actual H versus simulated mean H shown in Table 4 confirms the robustness of the replication model as we are able to generate a random realization of price path with Hurst estimation that precisely matches the characteristic H of the original time series for all five commodities.

Again, for the sake of consistency, the stopping local time is fixed at $L_m(t) = 10$, where the reference level m is set to be equal to the mean value of each time series obtained from the simulation. Figure 10, 11, and 12 below show the dynamics of three excursion statistics as we move from the commodity with lower H to the one with higher H. As expected, we see similar behavior compared to the case of the synthetic fBm time series. Specifically, the overall local time exhibits a downward sloping trend as we goes from ergodic to persistent time series. The stopping natural time is an increasing function in Hurst exponent. However, it is worthwhile to note that in Figure 10 we do not see the result we would generally be seeing with synthetic data where we normalized the starting price of all time series to a base of 100. In this particular case, the mean reference level for wheat contract seems to be out of sync with the rest of the commodities. The reason why it is higher than it ought to be is because the absolute price level of wheat futures is much higher than the other four contracts. This implies that the dynamics of the excursion reference level is actually dependent on the underlying price level of the assets. Hence, when dealing with real market data where there is significant difference in price level among the securities concerned, one must be aware of this limitation and adjust for it accordingly.

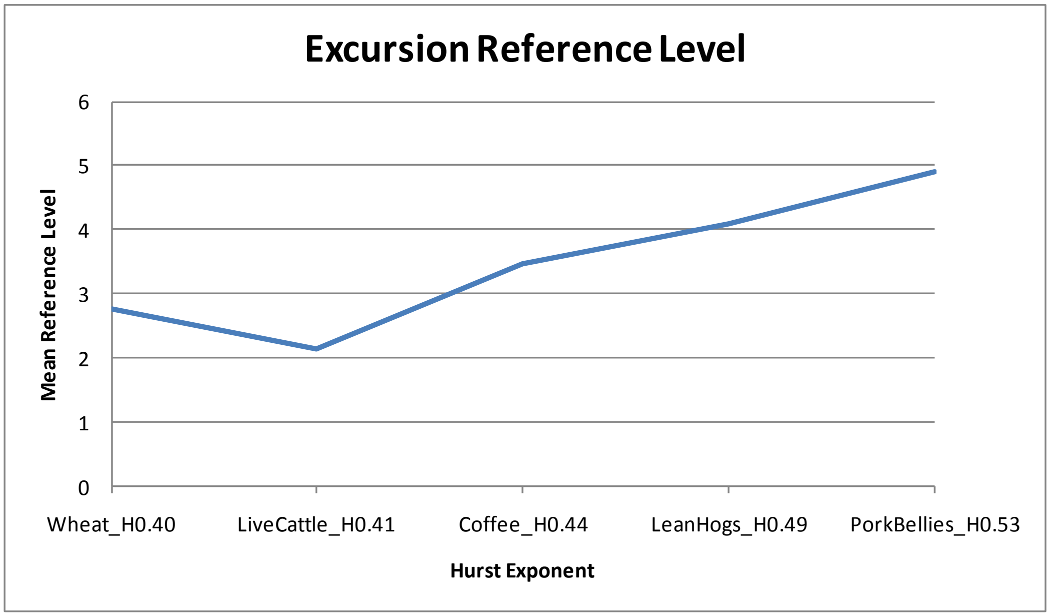


Figure 10: Dynamics of the excursion reference level for commodity futures price data

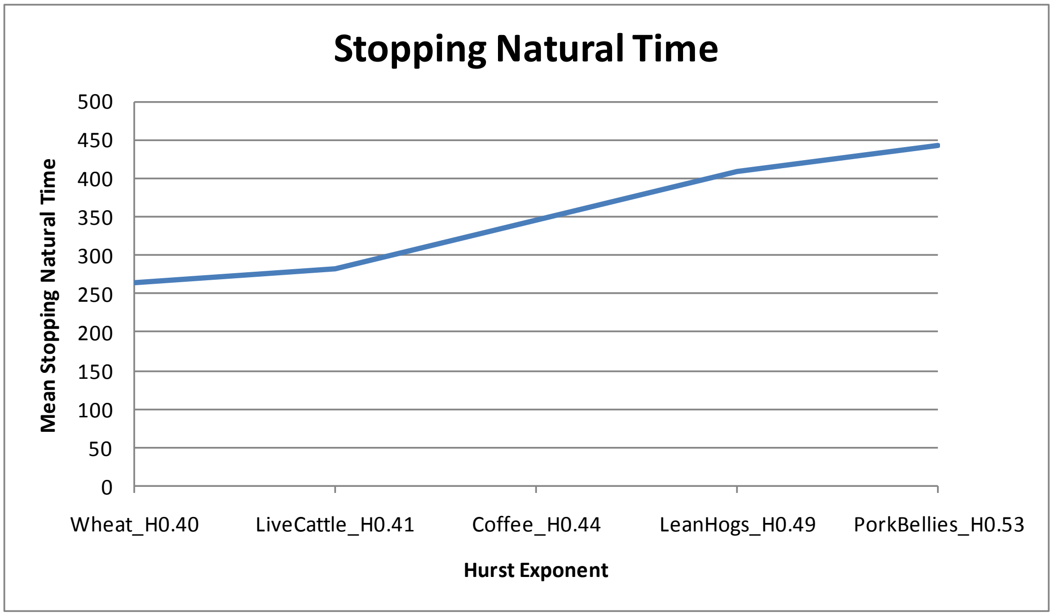


Figure 11: Dynamics of the stopping natural time for commodity futures price data

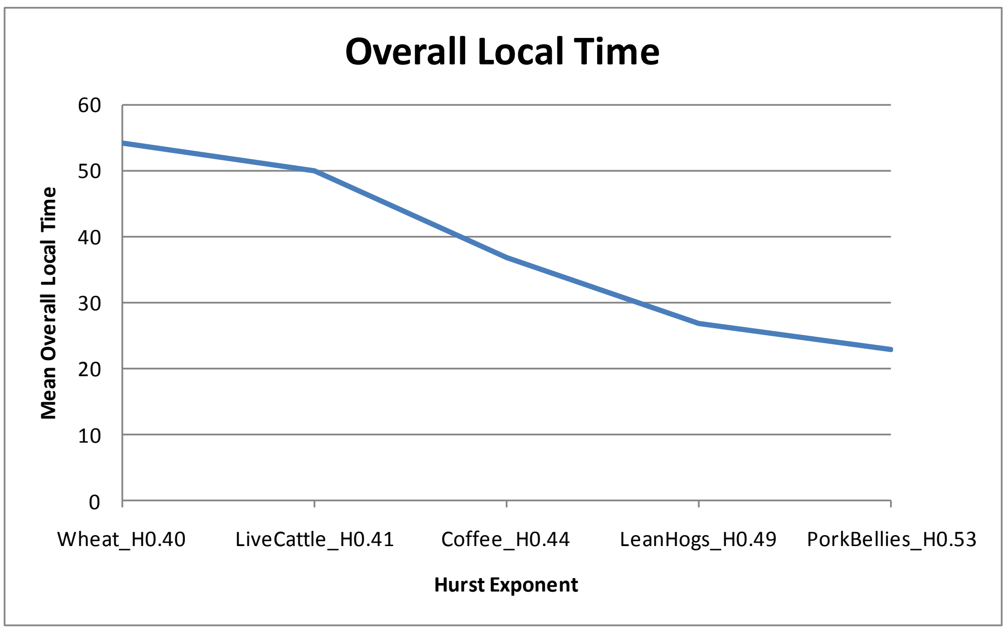


Figure 12: Dynamics of the overall local time for commodity futures price data

Finally, we collect the mean excursion measure at each excursion length (bin) for each of the five commodity futures. Then, for each bin, we rank the time series based on their resulting mean excursion measure. The commodity with the highest measure is then picked for each bin. Figure 13 illustrates this relationship graphically. Note that similar trend to what was shown in Figure 7 is revealed again here with the use of commodity future prices data, even with minimal variability in the degree of fractals among the five time series analyzed. The commodity with lower Hurst estimation achieves the highest mean count at shorter excursion length, but as we move into bin with higher excursion length the commodity with higher Hurst exponent clearly dominates in the limit.

From Figure 14, we can see again the rather smooth graphs showing the behavior of the processes through the mean count value across the bins. Notice that three out of five commodities exhibit the decaying structure we would expect as shown in the case of synthetic data. However, wheat and live cattle contracts, with mean Hurst exponent equals 0.40 and 0.41 respectively, show a somewhat interesting result. At 'bin 48', both commodities experience a mean count that is around three times more than what it ought to be based on the excursion theory. It is clearly not a systematic or programming error as it only happens to two out of five commodities. It is interesting because both processes are somewhat fractal with similar mean Hurst exponent estimation, and the outlier exists at the same excursion length of 48-day excursion which is roughly equal to the length of time between two consecutive futures contracts. This raises a few questions which might be worth investigating in the future. First, could this be an evidence of some sort of market manipulation? This would be an interesting path to explore within the area of forensic finance. Second, is it just a characteristic of a particular contract or market? Regardless of the answers to these two questions, this finding by itself is evidence that shows the potential usefulness of Itô's excursion theory as a tool to detect abnormal trading behavior in the asset market.

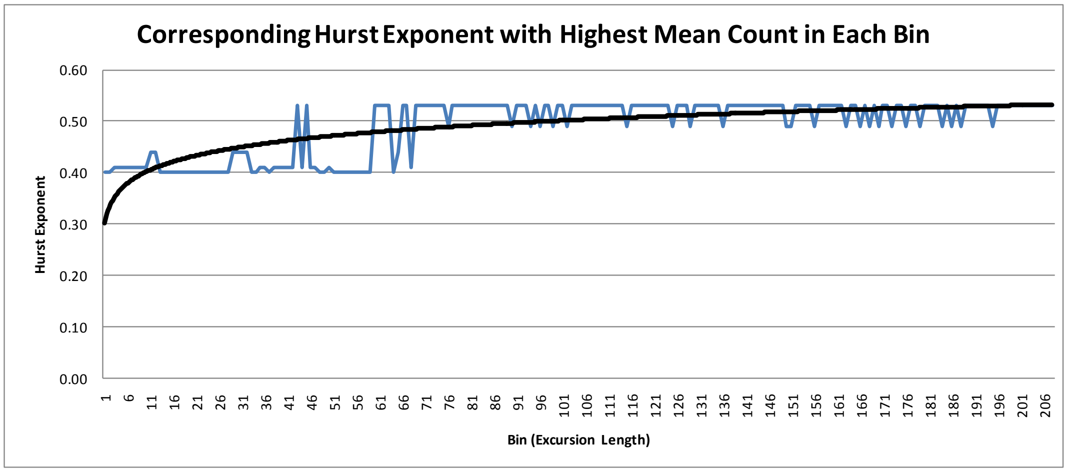


Figure 13: Relationship between Hurst exponent and mean excursion measure for commodity futures price data

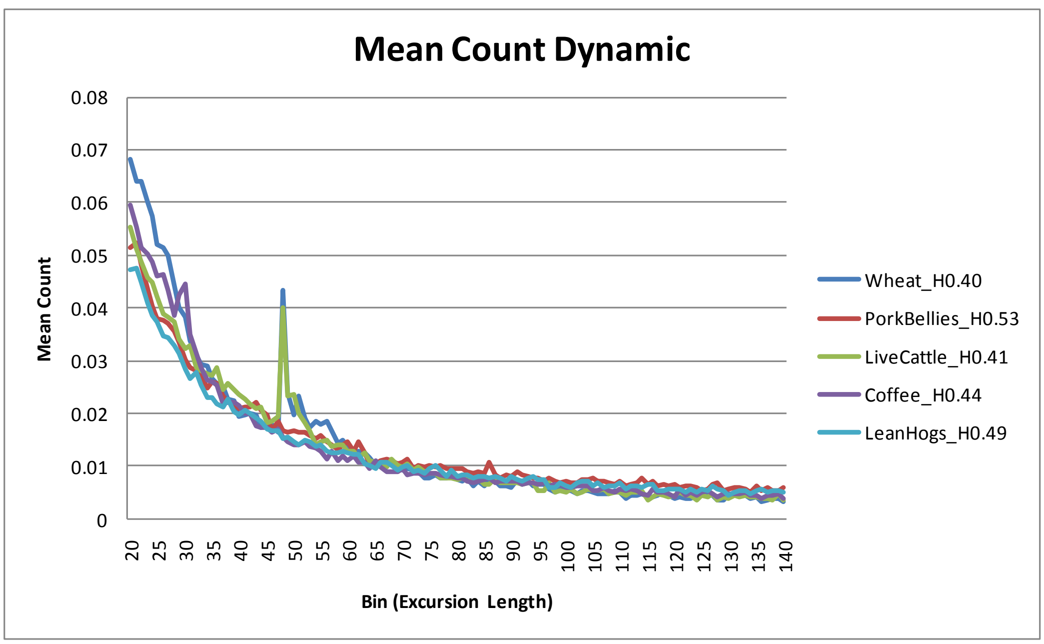


Figure 14: Mean count dynamics for five commodity futures

5. Conclusion

Mathematical finance and econometrics are not the easiest and most readable subjects to work with. Itô's excursion theory, on the contrary, allows for a powerful, yet intuitive tool to analyze complex time series data. It provides enormous clarity through graphical representation used to simplify mathematically involved stochastic processes. The excursion statistics and measure associated with the theory are easy to compute, and they are shown to be useful in several quantitative and probabilistic analyses related to the true nature of the original process.

An attempt to empirically link the Itô's excursion theory to some practical applications via Hurst exponent, a measure that describes the degree of fractal in a generalized fractional Brownian motion process, is successfully implemented in this paper using both synthetic fBm and commodity futures price data. The original time series are replicated using Monte Carlo simulation technique with proprietary algorithm developed in the previous study. With an extensive set of resulting data collected from the simulation, the theory is verified by observing the excursion measures and their corresponding fitted distribution. The excursions-valued process is shown to follow a binomial distribution which is a robust substitute for Poisson distribution as suggested from the theory. The result also shows that a process with low Hurst exponent or short-memory process has higher mean excursion measure at low excursion length as compared to a process with high Hurst exponent or a long-memory process. On the other hand, we see systematic wandering with longer excursion in a long-memory process with Hurst exponent higher than 0.5

It is very promising that combining the measures suggested from the Itô's excursion theory and Hurst exponent estimation together would lead to a unique tool that could be used to study the behavior of asset price returns in financial markets. The prediction power obtained from the method could be used by investors and traders to guide them in timing their investment. Regulators would benefit from the ability to detect abnormal market behaviors that are not consistent with the random walk or efficient market hypothesis and act accordingly to reduce market volatility and turbulence. This study opens up a whole new arena for practical applications that could potentially benefit the financial community at large.

REFERENCES

Abry, P. (1996). The wavelet-based synthesis for fractional Brownian motion proposed by F. Sellan and Y. Meyer: Remarks and fast implementation, Applied and Computational Harmonic Analysis. **3:** 377-383.

Black, F. and M. Scholes (1973). The pricing of options and corporate liabilities, Journal of Political Economy. **81:** 637-654.

Campbell, P. and S. Abhyankar (1978). Fractals, form, chance and dimension, Springer. **1:** 35-37.

Carmona, P. and L. Coutin (1998). Fractional Brownian motion and the Markov property, Laboratoire de statistique et probabilités, Université Paul Sabatier.

Fama, E. F. (1970). Efficient capital markets: A review of theory and empirical work, The Journal of Finance. **25:** 383-417.

Itô, K. (1972). Poisson point processes attached to Markov processes, Proceedings of the Sixth Berkeley Symposium on Mathematical Statistics and Probability. **3:** 225-239.

Lo, A. W. (2007). Efficient markets hypothesis. The New Palgrave: A Dictionary of Economics. S. D. L. Blume, eds., Palgrave Macmillan Ltd.

Malkiel, B. G. (2003). The efficient market hypothesis and its critics, The Journal of Economic Perspectives. **17:** 59-82.

Mandelbrot, B. B. (1982). The fractal geometry of nature, Freeman.

Mandelbrot, B. B. and J. W. Van Ness (1968). Fractional Brownian motions, fractional noises and applications, SIAM Review. **10:** 422-437.

Merton, R. C. (1973). Theory of rational option pricing, The Bell Journal of Economics and Management Science. **4:** 141-183.

Peters, E. E. (1989). Fractal structure in the capital markets, Financial Analysts Journal. **45:** 32-37.

Peters, E. E. (1996). Chaos and order in the capital markets: a new view of cycles, prices, and market volatility, John Wiley & Sons Inc.

Pitman, J. and M. Yor (2007). Itô's excursion theory and its applications, Japanese Journal of Mathematics. **2:** 83-96.

Rogers, L. C. G. (1989). A guided tour through excursions, Bulletin of the London Mathematical Society. **21:** 305-341.

Shreve, S. E. (2004). Stochastic calculus for finance: Continuous-time models, Springer Verlag.

Turvey, C. G. (2007). A note on scaled variance ratio estimation of the Hurst exponent with application to agricultural commodity prices, Physica A: Statistical Mechanics and its Applications. **377:** 155-165.

Turvey, C. G. (2010). Biography: Kiyosi Itô and his influence on the study of agricultural finance and economics, Agricultural Finance Review. **70:** 5-20.

Watanabe, S. (2010). Itô's theory of excursion point processes and its developments, Stochastic Processes and their Applications. **120**: 653-677.

Weron, R. (2002). Estimating long-range dependence: finite sample properties and confidence intervals, Physica A: Statistical Mechanics and its Applications. **312:** 285-299.

CHAPTER IV

HURST TRADING: A FRACTAL APPROACH TO MOMENTUM-BASED TRADING STRATEGY

1. Introduction

The concept of momentum trading has been introduced and studied rigorously in various academic papers and also exploited by many traders in several markets (Jegadeesh and Titman 1993; Zhang 2010). Similar to any proprietary trading strategies, the profit generated from momentum style of trading has diminished over the past decade as more and more people have adopted the strategy in their trading activities. One might argue that the huge influx of momentum arbitrageurs has brought the market back to the efficient level. We think it is time for new innovation.

In this paper, we extend the concept of traditional momentum trading to the fractal dimension by incorporating the use of the Hurst exponent estimation in order to come up with a more precise trading signal. This would allow one to better time the entry/exit points and thus be able to take advantage of both momentum and reversal in the markets. The ability of our model to predict reversal using the knowledge from the Hurst exponent parameter that characterized the time series illustrates how our approach is more superior and elegant than the traditional method.

The paper is organized as follow. In Section 2, we describe traditional momentum style trading and introduce the concept of Hurst exponent as a time-varying gauge for market efficiency. Then, in Section 3, we outline our research methodology. This involves the introduction of a rule-based statistical arbitrage trading strategy, which we called "Hurst Trading", that integrates traditional momentum trading with Hurst exponent. As a point of comparison, we also present in this study a "Buy and Hold" style of trading. In Section 4, we employ a series of synthetic fractal processes with varying Hurst exponent as an input in our trading model. We show the distribution of profits and losses generated from a simple market-neutral, Hurst Trading algorithm across time series with different Hurst exponent as well as several other interesting output variables. The results are strongly convincing that should the market deviate from being random as characterized by having a Hurst exponent equal to 0.5, consistent profit could be made by just following this simple trading strategy. For this particular set of synthetic time series, it is worth noting that the strategy seems to work exceptionally well for mean-reverting process with Hurst exponent smaller than 0.5 but not so true for process with high Hurst exponent. To test the null of random and efficient market, we further apply the Hurst Trading strategy to real stock price data in various stock markets and again run a backtesting. We provide comparison in profits and losses generated from three types of trading strategies namely: Hurst Trading, traditional momentum trading, and Buy and Hold style. The Hurst Trading algorithm is shown to outperform the other two alternatives by a wide margin whenever the market is characterized by a Hurst exponent that is significantly different from 0.5, a fractal process. This is especially true among small-cap stocks in the Russell2500 Index.

2. Momentum trading with Hurst exponent

The ability to forecast the price movement in the stock markets is no doubt an extremely valuable skill that every trader would hope to acquire. Successful timing and prediction of the market would lead to enormous profit. Given the high stakes involved, market participants have been trying to come up with tools that would help them beat the market. This motivation eventually leads to the discovery of momentum effect in stock returns.

2.1 Traditional momentum trading

Several studies have documented the existence of momentum effect in stock returns. (Jegadeesh and Titman 1993) found that a zero-cost momentum strategy of buying past winners and selling past losers generates significant average profits for stocks traded on the NYSE and AMEX using post-1940 data. (Rouwenhorst 1998) saw similar evidence for stocks traded on the European exchanges. On the other hand, (Hameed and Kusnadi 2002) found no evidence of significant profits from unrestricted momentum strategies in Asian markets. These studies used monthly stock return data. Moreover, there is growing criticism on the evidence of predictable patterns found from the studies that support momentum strategy as it could suffer from data-snooping biases since most of these studies examine the same set of data (Lo and MacKinlay 1990; Foster, Smith et al. 1997). Overall, the literatures yield a mixed impression on the validity of possible excess return generated from momentum trading strategy.

To avoid being trapped in this cycle of criticism altogether, we build the momentum trading strategy from the ground up by resorting to the basic definition of momentum effect. Momentum effect can be classified as a phenomenon whereby the stocks that have outperformed (underperformed) the average stock return over the last several observation periods tend to perform better (worse) than the average stock return over the next few periods. This immediately suggests a trading strategy of buying past winners and selling past losers. To be precise, in this study we define traditional momentum trading strategy as followed.

If the 5-Day Moving Average return is positive, then BUY.

If the 5-Day Moving Average return is negative, then SELL.

We use 5-Day Moving Average in order to capture the short-term momentum effect.

2.2 Hurst exponent

In the field of fractal time series and chaos theory, Hurst exponent, ranging from 0 to 1, emerges as a parameter that describes the degree of fractality in any stochastic processes. For $H = 0.5$, the process becomes a standard Brownian motion where the increments are independent. If $H < 0.5$, the increments are negatively correlated resulting in a mean-reversion or ergodic process. When $H > 0.5$, they are positively correlated and lead to a long-memory process (Bassingthwaighte and Raymond 1994; Carmona and Coutin 1998; Alvarez-Ramirez, Cisneros et al. 2002; Turvey 2007; Biagini, Hu et al. 2008).

In this paper, we will use the scaled variance ratio technique from (Cannon, Percival et al. 1997; Turvey 2007) that is quite distinct but consistent with *R/S* analyses (Hurst 1951; Mandelbrot and Van Ness 1968) to estimate the Hurst exponent of a time series process:

$$\frac{E\left[Y(t+k)-Y(t)\right]^2}{E\left[Y(t+1)-Y(t)\right]^2}=\frac{\sigma_k^2}{\sigma_1^2}=\left(k\right)^{2H}, \quad (1)$$

where Y_t is the log price at time t. We can simply solve Eq. (1) for the value of H once we figured out the variance of the return series at lag k and lag 1.

The concept of Hurst exponent is fairly new and only a handful number of studies have tried to exploit its power in forecasting predictability in a time series data. (Batten and Ellis 1996) studied the fractal structures of the daily logarithmic returns on the spot USD/JPY exchange rate for the period from 3 March 1987 to 8 September 1993 and found that there was persistence in the time series during that sample period in favor of the continued depreciation trend in USD (H~0.59). They claimed that following a naïve buy-and-hold trading strategy on JPY would result in excess return during that period, but did not provide systematic trading algorithm to arbitrage the market in general. (Ivanova and Wille 2002) presented some dynamical analysis of the S&P500 momentum using Moving Average, Hurst exponent, and traditional charts/tools in technical analysis, but did not mention about the possible, precise trading strategy that can be implemented using these measurements. (Alvarez-Ramirez, Cisneros et al. 2002) applied multifractal Hurst analysis to the crude oil market and found that crude oil prices exhibit long-memory effect. However, no trading strategy was mentioned. Recently, (Eom, Choi et al. 2008) used Hurst exponent as a measurement of the degree of market efficiency and showed using nearest-neighbor prediction method that high Hurst exponent corresponds to high degree of prediction or high hit rate. This work was based on empirical results from 60 market indices from around the world using monthly data. Again, no explicit trading strategy was described to take advantage of this knowledge.

3. Rule-based statistical arbitrage trading strategy

In this study, we attempt to form an explicit trading strategy to arbitrage the markets given the knowledge we have on the Hurst exponent estimation and the return time series of the data. The goal is to extract the information needed to time the correct entry/exit points in the market. Based on a set of rules, the trading strategy is to be implemented with high discipline over a certain time horizon. A thorough understanding of the strengths and weaknesses of the trading algorithm is crucial to make it profitable and successful.

3.1 Hurst exponent evolution

Hurst exponent can be thought of as a time-varying gauge for market efficiency. The estimation of the Hurst exponent for a particular time series will change as time goes by. Efficient market hypothesis demands that the Hurst exponent be equal to 0.5 which implies a random process. However, based on empirical results that we will show shortly, the Hurst exponent that characterizes the price series of a particular stock does deviate around that critical level. This means that even though the process might be random over a longer time horizon, there is a certain level of short-term predictability due to this fluctuation in Hurst exponent.

Table 1: Extracting trend and reversal dynamics from Hurst exponent

		5-Day MA Hurst Exponent			
		>0.5 & >30MA	**>0.5 & <30MA**	**<0.5 & <30MA**	**<0.5 & >30MA**
30-Day MA Hurst Exponent	**>0.5 & >5MA**		R	RRR	
	>0.5 & <5MA	T			
	<0.5 & <5MA	TT			R
	<0.5 & >5MA			RR	

T = weak trend
TT = strong trend
R = weak reversal
RR = moderate reversal
RRR = strong reversal

Table 1 illustrates a combination of possible range of value for 30-Day Moving Average Hurst exponent and 5-Day Moving Average Hurst exponent of a particular time series. To achieve the best result with regard to market timing, we will only focus our trading effort on the two extreme cases which are TT and RRR. Moreover, trading signal strength can be quantified and assigned numerical value by taking the difference between the 5-Day MA and 30-Day MA Hurst exponents.

$$(5MA - 30MA) = \begin{cases} large\ positive, & Trend \rightarrow TT \\ large\ negative, & Reversal \rightarrow RRR \end{cases}$$

3.2 Returns evolution

Along with the information from the evolution of the Hurst exponent over time, we can incorporate to it the returns patterns over the past few observation periods to help in guiding buy and sell signals for the strategy. Here, returns are defined as the logarithmic returns or the difference between the natural logarithm of prices for two subsequent periods. In order to capture short-term effect, we look at the 5-Day MA returns.

Table 2 describes the signal synthesis, integrating information from the Hurst exponent estimation and the returns evolution. The four possible scenarios are buy at bottom, buy on trend, sell at peak, and sell on trend.

Table 2: Generating buy and sell signals from Hurst exponent and returns dynamics

		Hurst Exponent	
		TT	**RRR**
5-Day MA Returns	**Positive**	BB	S
	Negative	SS	B

B = buy at bottom
BB = buy on trend
S = sell at peak
SS = sell on trend

3.3 Hurst Trading – A Simple Algorithm

This simple trading algorithm which we called "Hurst Trading" takes advantage of the information extracted from the Hurst exponent and the returns evolution of an asset's price series as described earlier. One would buy (sell) when there is B/BB (S/SS) signals. After that hold on to the position until the first S/SS (B/BB) signals appear, and then take appropriate action to close out the position. Subsequently, the trading process starts over again. Notice that we only hold one position at a time for this simple algorithm. There is no accumulation of position in one stock; this is a safety feature that prevents severe loss of capital that could lead to margin call in the case that the algorithm fails to perform in any unforeseeable circumstances.

3.4 Buy and Hold style

As a point of comparison, we also present in this study a "Buy and Hold" style of trading. It simply represents passive trading style. We define this trading style as followed.

BUY at the start of the trading period.

SELL at the end of the trading period.

All returns in this study are calculated in annualized term based on the formula below.

$$\frac{(S-B)}{B} \times \frac{252}{n}, \qquad (2)$$

where S = sell value

B = buy value

n = the holding period between the buy and sell transactions in trading days.

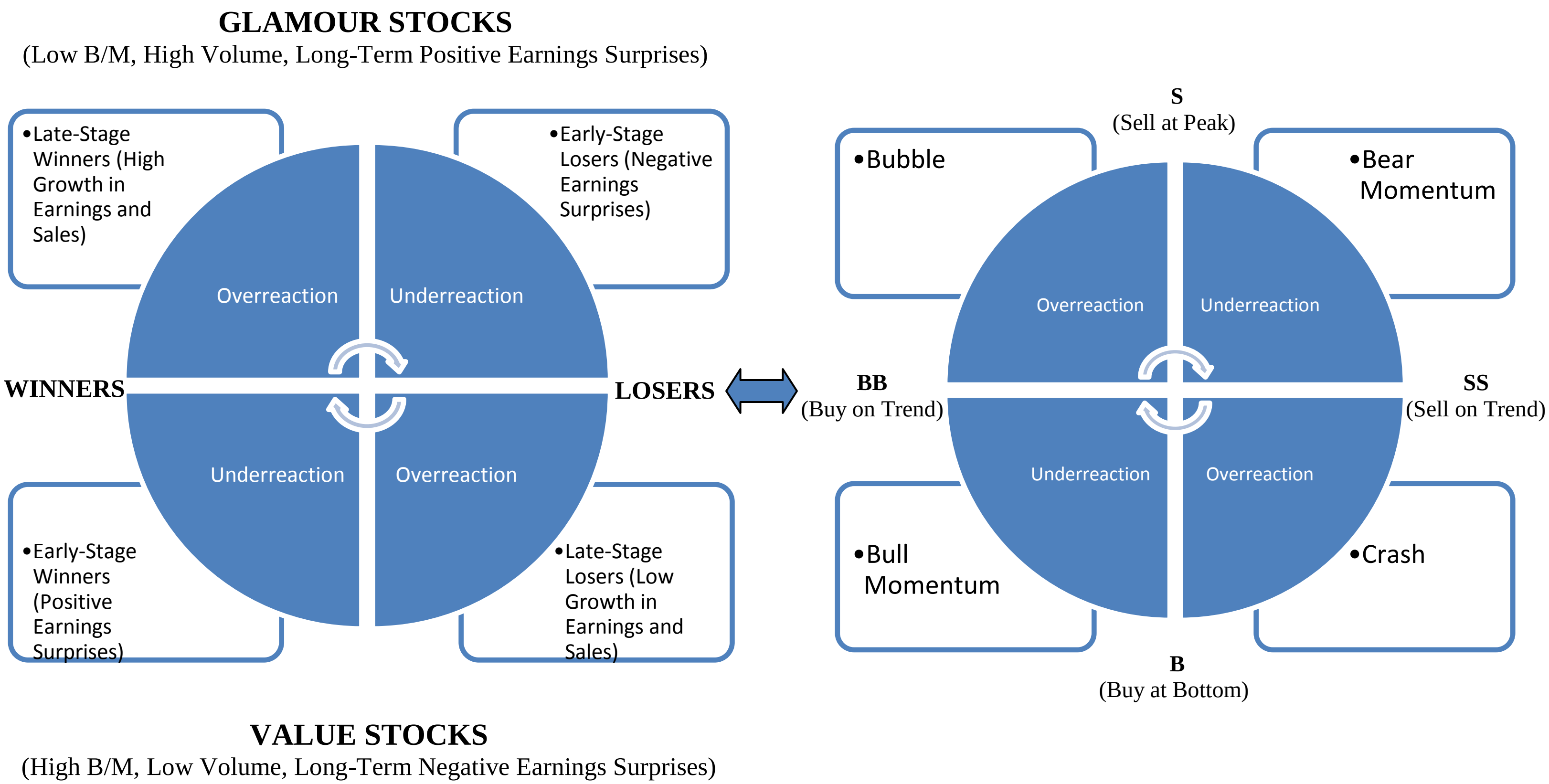


Figure 1: An analogy between Momentum Life Cycle by (Lee and Swaminathan 2000) (Left) and the motivation behind the concept of Hurst Trading (Right)

4. Results

In this section, we discuss the empirical results for three different trading strategies namely: Hurst Trading, traditional momentum trading, and Buy and Hold style. The presentation is divided into two parts. In Section 4.1, results are shown for synthetic fractal data. In Section 4.2, we present empirical results on actual stock data in various Large-Cap and Small-Cap indices.

4.1 An evidence from synthetic fractal data

Some known, synthetic time series with Hurst exponent ranging from 0.1 to 0.9 are generated in Matlab using the algorithm proposed by (Abry 1996) in order to observe and compare the performance of each trading strategy as we vary the underlying characteristic of the asset's price process. This would allow us to understand and validate the skill embedded in each trading strategy under controlled environment. Each time series contains 2150 data points.

4.1.1 Backtesting results

Figure 2 shows the annualized returns generated from each of the three trading strategies when applied on a set of synthetic fractal data with varying Hurst exponent. From the chart, we can see that the Hurst Trading strategy produces the highest return among all three strategies for process with Hurst exponent ranging from 0.1 to 0.5. Returns are decreasing from around 150% per annum to zero in this range of Hurst exponent. For process with Hurst exponent larger than 0.5, the Buy and Hold trading style yields the best results. Traditional momentum trading style appears to be the least performing strategy among the three strategies under study here.

From Figure 3, we could see that trading based on Hurst Trading strategy requires fewer turnovers than the traditional momentum trading style when observed over the same investment horizon across all time series with Hurst exponent ranging from 0.1 to 0.9. This would imply that trading based on Hurst Trading strategy actually subjects the trader to lower transaction costs while generating higher return, making it more attractive and superior than the traditional momentum trading style. This is a result of an extra layer of filter provided by the Hurst exponent that only triggers buy and sell signals when it is optimal to do so. Moreover, notice the inverted v-shape graph for the number of transactions as we vary the characteristic Hurst exponent of the time series. This indicates that as the process becomes more predictable as characterized by the deviation of Hurst exponent from the 0.5 level, the Hurst Trading algorithm can perform better even with a fewer number of turnovers. On the other hand, the turnovers resulted from trading with traditional momentum style is almost linearly decreasing in Hurst exponent. It trades less often when there is positive persistence in the time series and more often when the process is mean-reverting. Figure 4 provides more description to the frequency of trading based on these two trading style. It describes the average holding period in number of trading days. We see longer holding period as expected when the returns are highly positively correlated, signifying momentum play.

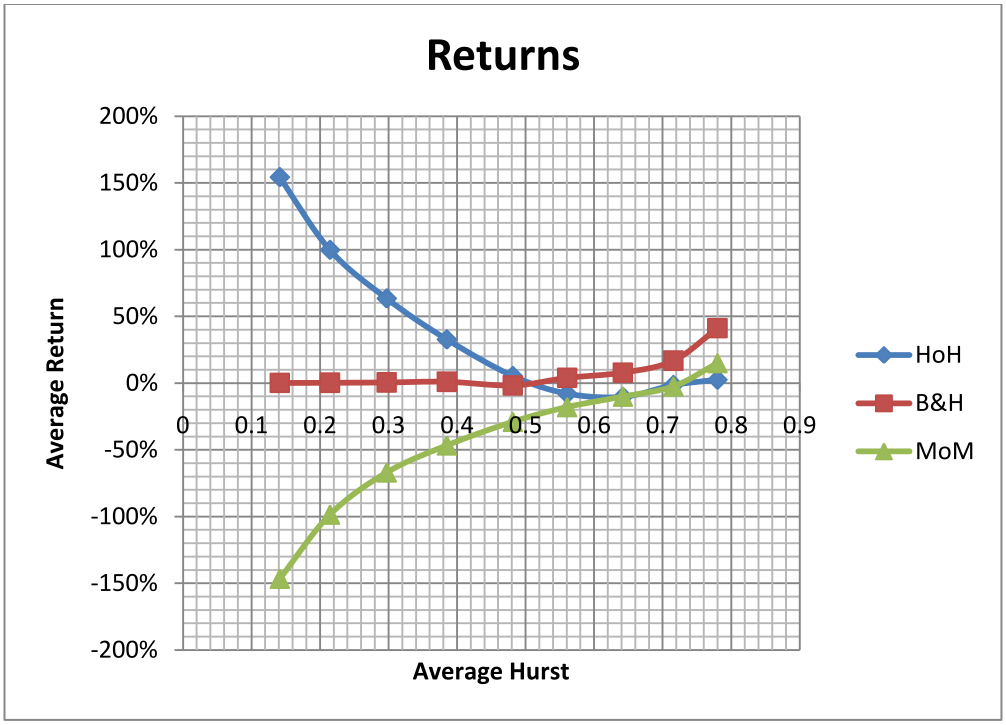


Figure 2: Comparison among annualized returns from 3 different trading strategies

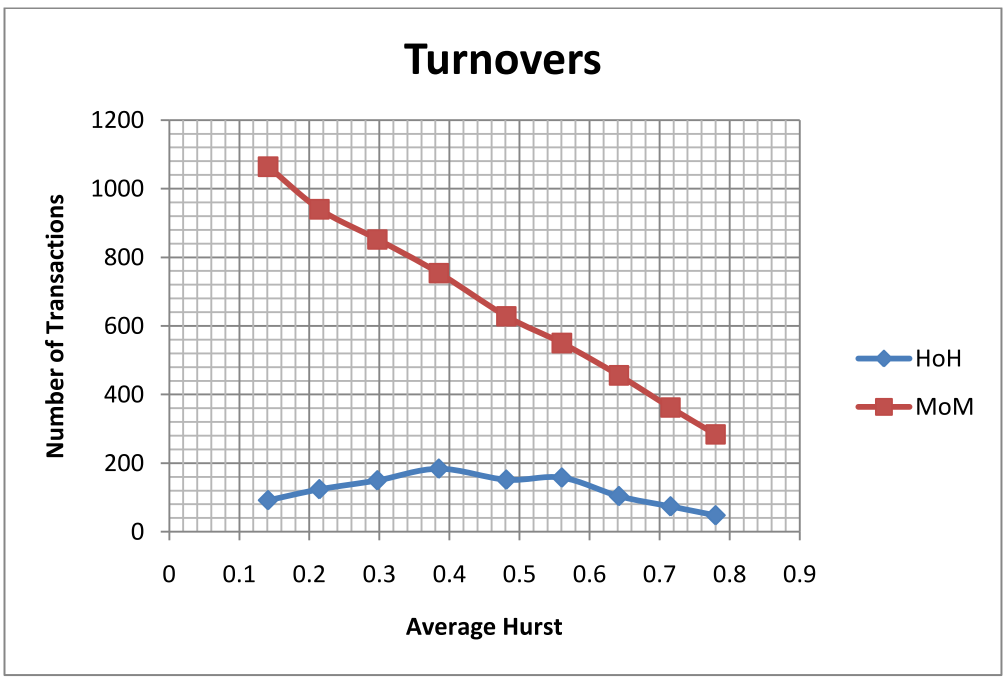


Figure 3: Difference in turnovers between Hurst Trading and traditional momentum trading strategies

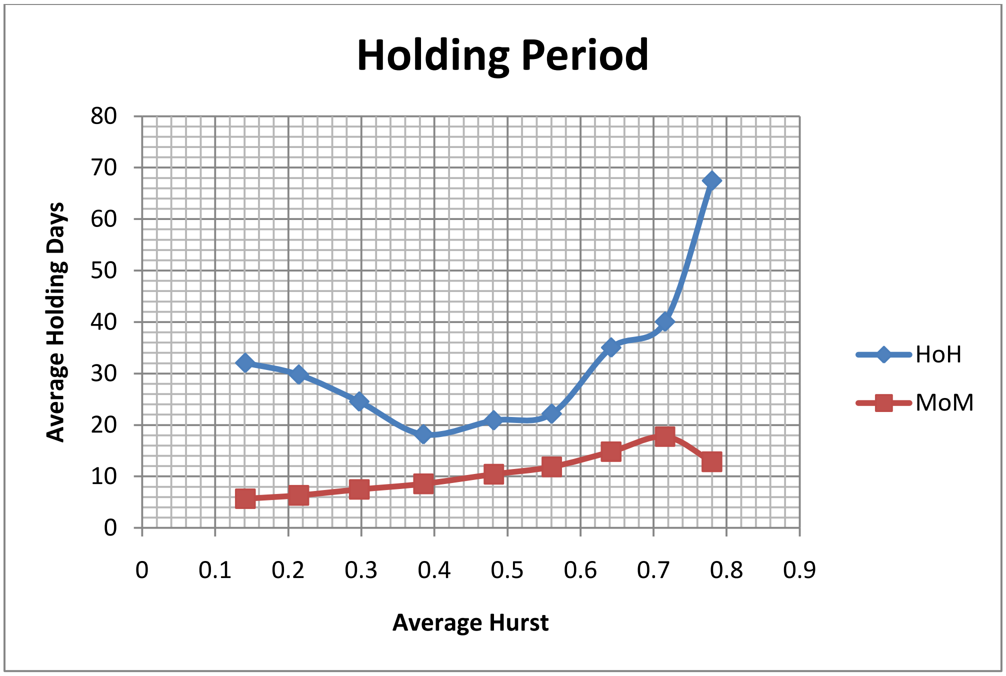


Figure 4: Difference in holding period between Hurst Trading and traditional momentum trading strategies

4.2 An evidence from stock market data

To test the null of random and efficient market, we further apply the Hurst Trading strategy to real stock price data in various stock markets. The data are taken from Bloomberg and consisted of the following:

- Large-Cap Index: Stocks in the Dow Jones Industrial Average (DJIA) Index and S&P 500 (SPX) Index. We use daily prices between 9/11/2002 – 3/24/2011.
- Small-Cap Index: Stocks in the Russell 2500 (R2500) Index. We use daily prices between 9/19/2002 – 4/1/2011.

Each stock price series contains 2150 sample points.

4.2.1 Backtesting results

Table 3 reports the proportion of stocks in each of the three indices that exhibits a certain value of average Hurst exponent in the study period. Our results agree with other studies in the field showing that most markets tend to exhibit memory within the range of H = [0.3,0.7] only (Opong, Mulholland et al. 1999; Cajueiro and Tabak 2004; Turvey 2007; Eom, Choi et al. 2008). All of the stocks in the DJIA Index have a Hurst exponent in the range between 0.4 – 0.5 indicating on average a random and efficient market. However, we find that stocks in the R2500 Index, with smaller market capitalization, tend to exhibit on average a Hurst exponent that could be significantly different from the value demanded by the random walk theory. This would potentially be the type of market that we want to arbitrage by implementing the Hurst Trading strategy.

Table 3: A breakdown showing the proportion of stocks in each index that belongs to a certain range of Hurst exponent

	0.0 - 0.1	0.1 - 0.2	0.2 - 0.3	0.3 - 0.4	0.4 - 0.5	0.5 - 0.6	0.6 - 0.7	0.7 - 0.8	0.8 - 0.9	0.9 - 1.0
DJIA	0.00%	0.00%	0.00%	0.00%	100.00%	0.00%	0.00%	0.00%	0.00%	0.00%
SPX	0.00%	0.00%	0.00%	1.35%	98.65%	0.00%	0.00%	0.00%	0.00%	0.00%
R2500	0.00%	0.00%	0.26%	13.30%	85.41%	1.03%	0.00%	0.00%	0.00%	0.00%

In this study, a simple portfolio with equal weight applied on each stock in the index is constructed. The results based on three different trading strategies are then collected and detailed statistics are shown in Table 4. The Hurst Trading algorithm is shown to outperform the other two alternatives by a wide margin. It generates on average a mean return of 69.86% per annum on the DJIA Index even though we saw earlier that all of the stocks in the DJIA Index have average Hurst exponent that is very close to 0.5. It was able to achieve this remarkable level of return by trading around the fluctuation in Hurst exponent estimation across time. Hence, in this case an active strategy like Hurst Trading with its unique skill is more superior than the passive trading style.

Table 5 reports the backtesting results based on a smaller universe of underlying stocks with Hurst exponent less than 0.4 only. We apply this filter based on Hurst exponent value to confirm the earlier result found with the synthetic data that as the Hurst exponent moves away from 0.5 the Hurst Trading strategy should be able to generate higher return. In fact, we find this to be true with real stock data as well since the return from the Hurst Trading strategy on the R2500 Index is shown to be 181.73% per annum compared to 116.78% per annum without the restriction. Moreover, the returns from the Hurst Trading algorithm are still far more superior than the other two strategies across all indices under study. This confirms our hypothesis that there are valuable skills and prediction power embedded in the Hurst Trading strategy.

To complete our discussion on the performance and feasibility of the Hurst Trading strategy, we also analyze the fluctuation in cashflow resulting from the implementation of the strategy across time. Table 6 reports the capital amount required to sustain the strategy as well as the maximum inflow of capital on a daily basis. For example, based on trading one share in any single name at a time, the maximum capital outflow or cushion required to sustain the Hurst Trading strategy on R2500 Index would only be around $10,000 on any given day. In contrast, the DJIA Index with much fewer constituents to trade would only see a maximum outflow of around $500 in a day should one decides to put on the Hurst Trading strategy in this 9-year period[1]. Figure 5, 6, and 7 illustrate the fluctuation in net cashflow position across time for the three index portfolios. This shows that one does not need to have a large amount of capital to trade based on this strategy and earn exceptional returns.

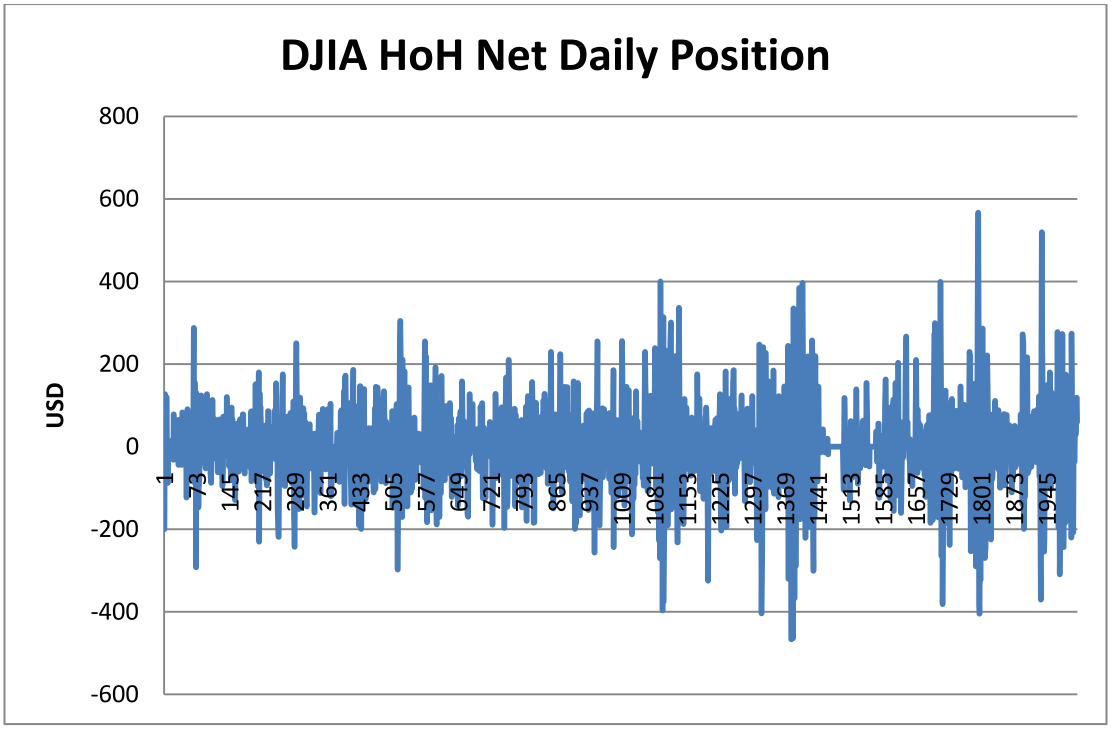


Figure 5: Fluctuation in the net daily cashflow position of the Hurst Trading portfolio on DJIA Index

[1] These figures are net amount on any trading day. In the calculations, we assume that a full proceed from the short sale is captured on the trade date itself.

Table 4: Summary statistics outlining a comparison of results across all three strategies (unrestricted universe of stocks)

	HoH				MoM				B&H					
	Mean Return	S.D. Return	Mean Turnover	S.D. Turnover	Mean Return	S.D. Return	Mean Turnover	S.D. Turnover	Mean Return	S.D. Return	Mean Average Hurst	S.D. Average Hurst	Mean Local Time	S.D. Local Time
DJIA	69.86%	60.18%	161.20	21.05	-124.21%	25.65%	676.20	20.43	9.98%	13.75%	0.45	0.02	43.03	27.03
SPX	86.82%	63.01%	158.88	23.49	-148.13%	40.56%	669.49	31.79	28.98%	63.50%	0.44	0.02	36.39	20.85
R2500	116.78%	105.46%	166.77	25.70	-184.07%	383.16%	687.21	48.88	27.91%	67.75%	0.44	0.04	40.39	25.91

Table 5: Summary statistics outlining a comparison of results across all three strategies (stocks with $H < 0.4$)

	HoH				MoM				B&H					
	Mean Return	S.D. Return	Mean Turnover	S.D. Turnover	Mean Return	S.D. Return	Mean Turnover	S.D. Turnover	Mean Return	S.D. Return	Mean Average Hurst	S.D. Averag e Hurst	Mean Local Time	S.D. Local Time
DJIA	N/A	N/A	N/A	N/A	N/A	N/A	N/A	N/A	N/A	N/A	N/A	N/A	N/A	N/A
SPX	91.09%	65.42%	148.00	25.08	-131.34%	34.75%	733.33	25.85	8.29%	2.98%	0.39	0.01	46.67	28.10
R2500	181.73%	87.32%	163.53	24.66	-134.94%	1033.12%	767.85	44.26	12.88%	31.52%	0.37	0.03	57.54	38.38

Table 6: Capital amount required to sustain the Hurst Trading strategy (unrestricted universe of stocks)

	Minimum Net Daily Cashflow	Maximum Net Daily Cashflow
DJIA	-$466.11	$566.15
SPX	-$5,166.99	$4,928.86
R2500	-$9,875.04	$9,763.53

* Note that returns are all presented in annualized term

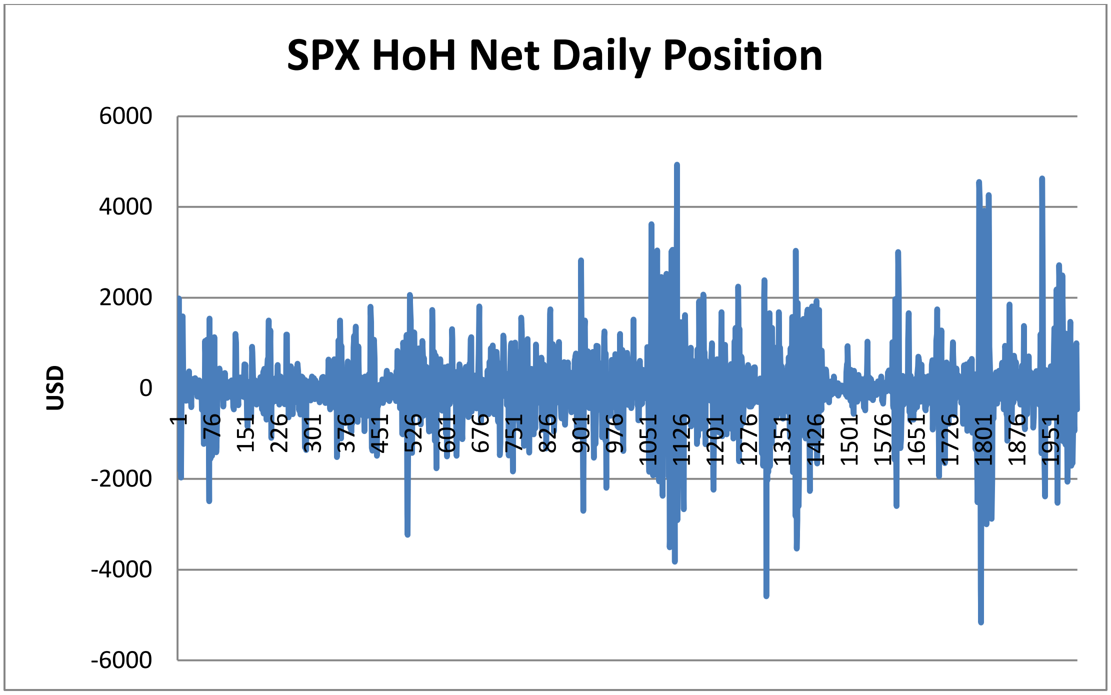


Figure 6: Fluctuation in the net daily cashflow position of the Hurst Trading portfolio on SPX Index

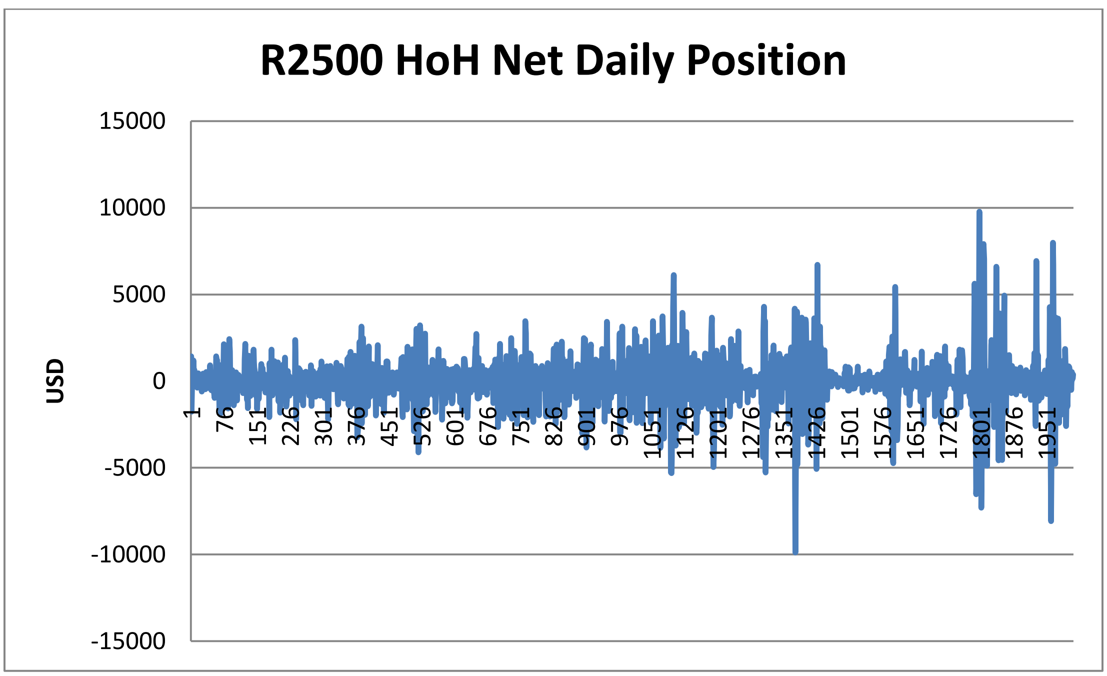


Figure 7: Fluctuation in the net daily cashflow position of the Hurst Trading portfolio on R2500 Index

5. Conclusion

Hurst Trading, which involves the application of Hurst exponent parameter to the traditional momentum trading strategy, is employed and tested on a set of synthetic fractal time series as well as on the actual stock price data for the period between 2002 to 2011. The Hurst exponent describes the fractal structure of the time series analyzed and is shown to fluctuate across time. The concept underlying the formation of the Hurst Trading strategy is to generate a more precise set of buy and sell trading signals by taking advantage of both momentum and reversal in the markets. Hence, this feature makes the strategy more attractive than the traditional momentum style since no prior knowledge of the persistency or structure of the traded market is required to make the program successful.

The results show that the Hurst Trading strategy is able to generate return in excess of 60% per annum on Large-Cap indices like the DJIA Index and the SPX Index during the sample period. It outperforms the traditional momentum trading style and the Buy and Hold style by a wide margin. The evidence is even stronger when applied to the R2500 Index, a Small-Cap index with average Hurst exponent in the range of 0.4 signifying a predictable, mean-reverting process. This further solidifies the idea that it is possible to arbitrage fractal markets characterized by a Hurst exponent that is significantly different from 0.5. The more fractal the process is, the higher the chance that the Hurst Trading algorithm would be able to correctly time the entry/exit points in the market.

REFERENCES

Abry, P. (1996). The wavelet-based synthesis for fractional Brownian motion proposed by F. Sellan and Y. Meyer: Remarks and fast implementation, Applied and Computational Harmonic Analysis. **3:** 377-383.

Alvarez-Ramirez, J., M. Cisneros, et al. (2002). Multifractal Hurst analysis of crude oil prices, Physica A: Statistical Mechanics and its Applications. **313:** 651-670.

Bassingthwaighte, J. B. and G. M. Raymond (1994). Evaluating rescaled range analysis for time series, Annals of Biomedical Engineering. **22:** 432-444.

Batten, J. and C. Ellis (1996). Fractal structures and naive trading systems: Evidence from the spot US dollar/Japanese yen, Japan and the World Economy. **8:** 411-421.

Biagini, F., Y. Hu, et al. (2008). <u>Stochastic calculus for fractional Brownian motion and applications</u>, Springer Verlag.

Cajueiro, D. O. and B. M. Tabak (2004). The Hurst exponent over time: testing the assertion that emerging markets are becoming more efficient, Physica A: Statistical Mechanics and its Applications. **336:** 521-537.

Cannon, M. J., D. B. Percival, et al. (1997). Evaluating scaled windowed variance methods for estimating the Hurst coefficient of time series, Physica A: Statistical Mechanics and its Application. **241:** 606-626.

Carmona, P. and L. Coutin (1998). <u>Fractional Brownian motion and the Markov property</u>, Laboratoire de statistique et probabilités, Université Paul Sabatier.

Eom, C., S. Choi, et al. (2008). Hurst exponent and prediction based on weak-form efficient market hypothesis of stock markets, Physica A: Statistical Mechanics and its Applications. **387:** 4630-4636.

Foster, F. D., T. Smith, et al. (1997). Assessing goodness-of-fit of asset pricing models: The distribution of the maximal R 2, The Journal of Finance. **52:** 591-607.

Hameed, A. and Y. Kusnadi (2002). Momentum strategies: evidence from pacific basin stock markets, The Journal of Financial Research. **25:** 383-397.

Hurst, H. E. (1951). Long-term storage capacity of reservoirs, American Society of Civil Engineers. **116:** 770-808.

Ivanova, K. and L. T. Wille (2002). Dynamical analysis of S&P500 momentum, Physica A: Statistical Mechanics and its Applications. **313:** 625-639.

Jegadeesh, N. and S. Titman (1993). Returns to buying winners and selling losers: Implications for stock market efficiency, The Journal of Finance. **48:** 65-91.

Lee, C. M. C. and B. Swaminathan (2000). Price momentum and trading volume, The Journal of Finance. **55:** 2017-2069.

Lo, A. W. and A. C. MacKinlay (1990). Data-snooping biases in tests of financial asset pricing models, Review of Financial Studies. **3:** 431-467.

Mandelbrot, B. B. and J. W. Van Ness (1968). Fractional Brownian motions, fractional noises and applications, SIAM Review. **10:** 422-437.

Opong, K. K., G. Mulholland, et al. (1999). The behaviour of some UK equity indices: An application of Hurst and BDS tests, Journal of Empirical Finance. **6:** 267-282.

Rouwenhorst, K. G. (1998). International momentum strategies, The Journal of Finance. **53:** 267-284.

Turvey, C. G. (2007). A note on scaled variance ratio estimation of the Hurst exponent with application to agricultural commodity prices, Physica A: Statistical Mechanics and its Applications. **377:** 155-165.

Zhang, Y. (2010). Two Essays on Momentum Strategy and Its Sources of Abnormal Returns. PhD diss., University of Tennessee.

CHAPTER V

CONCLUSION

This dissertation tackles the problem of non-normality in the distribution of returns and attempts to formulate a proprietary trading strategy to arbitrage the markets using appropriate statistical and mathematical tools. The first essay lays a foundation to understanding time series data by explaining the concept of unit root, autoregressive process, fractional Brownian motion, and its characteristic Hurst exponent. We outline a simple, yet robust and consistent method to generate fractal process using an autoregressive approach that is familiar to economists. The algorithm involves the implementation of stepwise regression and restricted least square method on a set of lag time series. We prove that having the sum of the lag coefficients equals unity in an AR(q) process is a necessary and sufficient condition for the existence of a unit root. On top of that, we also find that a simple autoregressive process with suitable lag coefficients is able to effectively replicate the fractal time series and preserves its characteristic Hurst exponent. Finally, we derive an equation that defines the relationship between the AR lag coefficients and the Hurst exponent that described a particular fBm process. This finding is crucially important since it allows one to understand how a choice of lag coefficients could affect the resulting behavior of the process generated from the model. It also allows one to solve for an unknown variable in the defining equation given the knowledge of the rest of the parameters. This gives researchers full control over the particular process they wish to generate.

Since information on the level of persistence in any given time series is embedded within the Hurst exponent, the second essay attempts to exploit the dynamics of such variable using measures invoked from the Itô's excursion theory. The theory allows for a powerful, yet intuitive tool to analyze complex time series data. It provides enormous clarity through graphical representation used to simplify mathematically involved stochastic processes. The excursion statistics and measure associated with the theory are easy to compute, and they are shown to be useful in several quantitative and probabilistic analyses related to the true nature of the original process. In the study, a set of original time series are replicated using Monte Carlo simulation technique adapted to the method proposed in the first essay. With an extensive set of resulting data collected from the simulation, we record the excursion measures and their corresponding fitted distribution. We find that the excursions-valued process actually follows a binomial distribution which is a robust substitute for Poisson distribution as suggested from the theory. Moreover, the empirical results also indicate that a process with low Hurst exponent has higher mean excursion measure at low excursion length as compared to a process with high Hurst exponent. On the other hand, we see systematic wandering with longer excursion in a long-memory process with Hurst exponent higher than 0.5.

Based on the discovery from the first two essays, the third essay combines these findings together to form a trading strategy called Hurst Trading with trading signals generated from the fluctuation in the dynamics of the Hurst exponent across time, among other indicators. Hurst Trading is an enhanced strategy to the traditional momentum trading strategy. The concept underlying the formation of the Hurst Trading strategy is to generate a more precise set of buy and sell trading signals by taking advantage of both momentum and reversal in the markets. Hence, this feature makes the strategy more attractive than the traditional momentum style since no prior knowledge of the persistency or structure of the traded market is required to make the program successful. We find that for the period between 2002 to 2011 the Hurst Trading strategy is able to outperform the traditional momentum strategy and the Buy and Hold strategy by a wide margin on stock trading in the DJIA Index, SPX Index, and R2500 Index. The return on the Hurst Trading strategy is in excess of 60% per annum on Large-Cap indices like the DJIA Index and the SPX Index during the sample period. The evidence is even stronger when applied to the R2500 Index, a Small-Cap index with average Hurst exponent in the range of 0.4 signifying a predictable, mean-reverting process. This further solidifies the idea that it is possible to arbitrage fractal markets characterized by a Hurst exponent that is significantly different from 0.5. In fact, the level of accuracy in timing the entry/exit points in the market of Hurst Trading is directly proportional to the degree of fractality in the time series data.